DOCUMENT RESUME

ED 077 714 SE 016 251

AUTHOR Rosskopf, Myron F., Ed.; And Others
TITLE Piagetian Cognitive-Development Research and Mathematical Education, Proceedings of a Conference (Columbia University, October 1970).
INSTITUTION National Council of Teachers of Mathematics, Inc., Washington, D.C.
PUB DATE 71
NOTE 243p.

EDRS PRICE MF-$0.65 HC Not Available from EDRS.
DESCRIPTORS *Cognitive Development; *Elementary School Mathematics; Geometric Concepts; Learning; *Learning Theories; Logic; *Mathematics Education; Number Concepts; Research; *Research Reviews (Publications)
IDENTIFIERS Proof (Mathematics)

ABSTRACT

Fourteen papers that were presented at an NCTM-sponsored conference on Piagetian research are included in this document. Topics covered include explanations of Piaget's developmental theory and descriptions of studies concerned with the development and growth of the mathematical concepts of proof, logical operations, functions, proportionality, probability, number and measurement, properties of numbers, and geometry. (DT)

ED 077714

Piagetian Cognitive-Development Research and Mathematical Education

National Council of Teachers of Mathematics

PIAGETIAN

Cognitive-Development Research

and

Mathematical Education

PIAGETIAN

Cognitive-Development Research and Mathematical Education

Proceedings of a conference conducted at Columbia University October 1970

Edited by

MYRON F. ROSSKOPF
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NATIONAL COUNCIL
OF TEACHERS OF MATHEMATICS

Library of Congress Catalog Card Number: 73-161876

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Preface

From very small beginnings during the early years of the 1960s, interest by mathematics educators in Piagetian research broadened until at several universities students were working on doctoral dissertations that clearly were almost as closely related to child-development psychology as to mathematical education. Uneasiness over little evidence of close cooperation between psychologists and mathematics educators led to persuading the National Council of Teachers of Mathematics to sponsor jointly with the Department of Mathematical Education, Teachers College, Columbia University, a conference on the Piaget type of research in mathematical education. Somewhat apart from New York City and with facilities for housing participants, the Greyston Conference Center of Teachers College was chosen for the site of the conference, held 18–23 October 1970.

The primary purpose of the conference was to promote more dialogue not only between mathematics educators but also between mathematics educators and psychologists. A daily schedule with much free time seemed an excellent way to encourage small groups interested in the same aspect of cognitive development to get together for discussions. Sometimes these were carried on during walks about the grounds of Greyston: sometimes a group gathered after dinner in someone's room, with the scheduled lectures serving as background information. The discussions led to new acquaintances and new understandings of both mathematics and child-development psychology.

A grant from the National Science Foundation made possible the papers that appear in this volume, the volume itself, and the participation of some sixty psychologists, mathematics educators, and doctoral students. All participants had a deep interest in Piagetian investigations, and all were enthusiastic about the opportunity to talk with fellow investigators.

Myron F. Rosskopf
Leslie P. Steffe
Stanley Taback

About the Authors

MILLIE ALMY is professor of psychology and education at Teachers College, Columbia University, where she also serves as principal adviser in the Program in Early Childhood Education. She is the author of several books and numerous articles dealing with child development and early childhood education. Her interest in the applications of Piaget's theory to problems of education began a decade ago when she became concerned with the paucity of intellectual stimulation in many of the kindergartens she observed. Seeking some means of appraising the children's level of thinking, she replicated some of Piaget's experiments and eventually conducted the two longitudinal studies on which her paper is based.

Although Dr. Almy has noted with satisfaction the increasing attention given during the past decade to the intellectual aspects of educational programs for young children, she believes that concern with the cognitive has sometimes been at the expense of the affective, expressive, and aesthetic. Her current work, largely exploratory in nature, deals with children's play, which she believes reflects all these aspects. Without an understanding of the role of play in the life of the young child, it seems to her impossible to plan adequately for his education.

HARRY BEILIN is a professor in the City University of New York Graduate Center and is active in both the developmental psychology and the educational psychology doctoral programs. He has previously served as head of each of these programs. Presently he is editor of the *Journal of Experimental Child Psychology*.

Dr. Beilin's area of specialization is cognitive development, and he has published widely in the field of children's language and cognition. His research has been supported for the past ten years by grants from the National Institute of Child Health and Human Development and the National Institute of Mental Health.

PETER DODWELL was educated in England, obtaining B.A., M.A., and D.Phil. degrees from Oxford University, the latter in 1958. Originally intending to become a mathematician, he was seduced away by an interest in logic and epistemology, so coming eventually to experimental psychology by way of philosophy.

Dr. Dodwell taught in the University of London, England, for three years before moving to Canada in 1958. Since that time he has been attached to Queen's University at Kingston, Ontario, with interludes at London University as a C. D. Howe Fellow and visiting professor, at the Center for Advanced Study in the Behavioral Sciences, Stanford, on a Guggenheim Fellowship for a year, and at Harvard for half a year as visiting professor of psychology. His recent research

has been in the field of perception, but he still retains an active interest in research on cognitive development in children.

KENNETH LOVELL is a graduate of the University of London and holds the degrees of B.S., M.A., and Ph.D. He taught in primary and grammar schools and in colleges of education, and he is now professor of educational psychology in the University of Leeds. He has written eight books and published nearly fifty papers.

One of Dr. Lovell's major interests is that of Piaget's cognitive-developmental system and its application to the analysis of children's understanding in the classroom—particularly in the fields of mathematics and science. From such analysis comes the belief that curriculum construction, and the quality of teaching, will be improved. It is his hope that the papers of this conference will help mathematics educators in two ways: (1) give them sufficient experience of differing experimental procedures to encourage them to set up their own research projects, and (2) help them to get their students, who are the future elementary and high school teachers, to make contact with their pupils in ways that will enable such teachers to recognize children's mathematical conceptions and misconceptions.

HERMINE SINCLAIR is professor of psycholinguistics at the University of Geneva and works in Piagetian theory in general and in the learning of cognitive structures in particular with Bärbel Inhelder and others. She is a coauthor of several books published by the Genevans and has published widely in the field of children's cognitive development. Currently in press is an account of investigations into the development of language in young children.

Dr. Sinclair studied at the University of Utrecht, concentrating on classical languages and historical linguistics. After World War II, travel, and marriage, she turned to the study of psychology at the University of Geneva. Her chance arrival in Geneva brought her to Piaget's psychology, and since that time she has been working at the Institut des Sciences de l'Éducation.

HENRY VAN ENGEN is a graduate of the University of Michigan, where he earned a Ph.D. in mathematics in 1935. After teaching in junior high schools and at Western Reserve University in the School of Education, he served as an assistant professor of mathematics at Kansas State University, Manhattan, and, from 1938 to 1958, as head of the Department of Mathematics at Northern Iowa University, then known as Iowa State Teachers College. In 1958 he was appointed to the position he now occupies as professor of education and mathematics at the University of Wisconsin, Madison. At various times he has served as editor of the *Mathematics Teacher*, as a member of the Commission on Mathematics of the College Entrance Examination Board, and as a member of the Board of Directors of the National Council of Teachers of Mathematics. He is presently chairman of the Editorial Panel of the *Arithmetic Teacher*.

Dr. Van Engen has long been interested in the analysis of meaning in arithmetic and in concept formation, having written major papers in these two areas. Among the mathematics educators in the United States, he was one of the first to become keenly interested in theories of cognitive development and their application to

mathematical instruction. Although he believes that research in cognitive development has major implications for mathematical education, he also believes the results of that research are not necessarily directly applicable to mathematical instruction. Much of his own research and that of his students deals with classroom applications of Piaget's theory. He thinks that in such research the mathematical educator must establish a more than superficial communication base with psychology while, at the same time, maintaining his own identity. Piaget's theory has a substantial mathematical content which must be understood apart from Piaget's psychology. It is here, Professor Van Engen believes, that mathematical educators can make their best contribution.

HERMINE SINCLAIR

Piaget's Theory of Development: The Main Stages

At first sight it would seem that a psychological theory that is regarded by its author as a "by-product" of his epistemological research and is therefore principally directed toward the investigation of knowledge and its changes in the history of mankind, as well as in the growing child, is ideally suited to educational applications. One of the aims of education is the fostering of knowledge, the endeavour to transmit to the next generation the experience of its forebears in the hope that the sum total of knowledge will be expanded. At the same time, for many different reasons, there is a general feeling that education is not good enough, that something should be changed, that not enough profit is derived from what is becoming a considerable number of years spent at school. It is thus not surprising that a number of educators have turned to Piaget's theory to seek help for new pedagogical approaches. Unfortunately, many of them have been disappointed; Piaget's theoretical approach has seemed too far removed from classroom reality. Recently, others have become very enthusiastic, seduced by the experimental situations Piaget has imagined, and there seems to be a regrettable tendency to take Piaget's problem situations and convert them directly into teaching situations.

Why I think this is regrettable is probably best explained by a metaphor: Piaget's tasks are like the core samples a geologist takes from a fertile area and from which he can infer the general structure of a fertile soil; but it is absurd to hope that transplanting these samples to a field of nonfertile soil will make the whole area fertile. A child's reactions to a few Piagetian tasks will enable a well-trained psychologist to give a fair description of that child's intellectual level; but teaching the solutions of these same Piagetian tasks to a group of children does not mean that

the children will thereby attain the general intellectual level of the child who can solve the tasks independently.

Many modern school programs emphasize doing and point out that seeing and hearing are not enough; such programs are sometimes called Piagetian, and, indeed, one of Piaget's basic principles is the primacy of action. However, this does not mean that children should spend all their early school years digging in sandpits and making mud pies, progressing to constructing buildings out of bricks and then to making systems of pulleys and levers.

—Educational applications of Piaget's experimental procedures and theoretical principles will have to be very indirect—and he himself has given hardly any indication of how one could go about it. His experiments cannot be modified into specific teaching methods for specific problems, and his principles should not be used simply to set the general tone of an instructional program.

It would thus seem necessary to study Piaget's theory as a whole before deciding which parts of it, if any, could be applied in the classroom. The theory is explicitly developmental and maintains that the explanation of the nature of adult knowledge is found by studying the way this knowledge has been built up; in other words, the adolescent explains the man, the child explains the adolescent, the toddler explains the child, and the infant explains the toddler. Even though all of you are familiar with parts of Piaget's work, a brief description of the psychological characteristics of cognitive development may be helpful. This will perhaps involve a certain amount of tedious repetition of facts known to all of you. My apologies!

According to Piaget, action, rather than perception, is the primary source of knowledge. To know objects, one has to modify them in some way—for instance, simply change their position. The main division into developmental stages is therefore based on the character of the actions that link the subject to the surrounding world.

From Sensorimotor Intelligence to Concrete Operations

A first period is called the sensorimotor stage. It is a preverbal period, or, speaking more generally, a period of direct action without representation. During this stage (lasting until about the middle of the second year) the world around the subject becomes more and more stable and organized. While at first the newborn baby seems to have no awareness of himself as distinct from the objects around him, by the end of this period he can perform the actions that assure the direct dependencies

between subject and objects. He has now acquired object permanency, a first cognitive invariant, as well as a first grouplike structure—the group of displacements, as Piaget has called it. In view of the paramount importance Piaget accords to actions that modify reality, it is no accident that his formalizations of the underlying cognitive structures are all in terms of groups of transformations—even at the level of sensorimotor intelligence. Constants, on the one hand, and action or operation structure, on the other, cannot be dissociated psychologically: at all stages of development they are no more than two sides of the same coin. However, it seems advisable to separate the invariants from their grouplike structures for the purposes of this brief sketch of the course of cognitive development toward the formal operations, which are the only ones that can be formally expressed in terms of group structures in the mathematical sense of the word *group*.

Constants

Let me start with the cognitive constants. Object permanency, achieved by the middle of the second year, means above all that the objects have now become "retrievable" or "retraceable"—the child no longer acts as if they disappear completely once they go out of his perceptive field; moreover, if they are hidden under several screens, he can recompose their successive displacements and find them again. Permanent objects are objects one can start "knowing"—they are no longer only objects to which one can react. A little later, objects acquire an identity that is no longer simply a function of the act of searching for them but that arises from the child's realization that several actions can be performed on the same object without its basic identity's being changed. For instance, a piece of wire can be twisted into the shape of a pair of glasses or scissors, but these different shapes do not alter the "sameness" of the piece of wire that was used to produce them. Although the child may put the glasses on his nose and pretend to look through them, he knows, and will say so if asked, that it is the same piece of wire that was used earlier for something else. The next step toward the establishment of quantitative constants is taken when the child begins to make a distinction between permanent and impermanent qualities of objects. The colour, suppleness, and material of the wire are permanent, but its shape is not. For children below the age of, say, seven, certain changes of shape imply a change in length. Nevertheless, the identity of objects has become more objective in the sense that it is now based on the objects' qualities rather than on the actions the subject performs on them.

The great novelty of the concrete-operational period is the change from qualitative identities toward quantitative constants. The first of these

quantitative constants is numerical conservation, which manifests itself in correct answers to the well-known questions about the numerical equivalence of two collections of objects. For an example, consider the experiment in which red and blue counters are used. Starting from two rows, one red and one blue, arranged in a visual one-to-one correspondence, the experimenter spreads out the blue counters so that they go beyond the limits of the red row. The child is then asked whether there are still just enough red counters to cover the blue ones, or whether there will be some blue counters left over, or whether there will not be enough, and so on. Before the age of five or six, the child will say that there are not enough red counters to cover the blue, that some blue ones will be left over, t. t there are more blue ones than red ones, and so on. But from five or six onward, the child affirms that the number has not changed and he can give arguments to explain his judgment: "You didn't add any"; "You can put them back like they were"; "They're just spread farther apart"; and so on.

Structures

We now come to the second aspect, that of the structure of actions: the action group of displacements, which is completely bound up with object permanency, slowly becomes elaborated during the sensorimotor period from very elementary, often hereditary action patterns. These action patterns (sucking, looking, grasping) at first form isolated entities, but soon they assimilate other objects (sucking thumbs, toys, etc.) and become coordinated (grasping *and* looking, etc.). Gradually, instead of several little isolated actions, there are more and finer coordinations, which culminate in an intended connection between a definite goal and the action sequence necessary for reaching that goal. During the first period of postsensorimotor but preoperational intelligence, the child starts to build up what Piaget calls a *semilogic*, that is to say, a logic of one-way mappings. In psychological terms, this means that the child understands that when one pulls the cord of a curtain, the curtain opens; the farther one pulls, the farther the curtain opens. These functional dependencies imply a real, physical link between cause and effect just as much as a conceptual dependency. Pulling (y) makes the curtain move (x), where $x = f(y)$; but you also have to know how far to pull to make the curtain go all the way back—knowledge of x depends on knowledge of y. These dependencies constitute a kind of semilogic, and their one-way character has been demonstrated in several studies.

In the following experiment, devised by N. van den Bogearts in 1968, the child is shown a toy truck which picks up counters in front of five different dolls. Each counter is put inside the truck in a line so that the

arrangement of counters mirrors the itinerary. The colours of the counters correspond to those of the dolls' dresses. The dolls are arranged on the table in a fixed pattern, but neither in a straight line nor in a circle. The first questions concern the arrangement of the counters inside the truck: "Which will be first?" "Which will be last?" "Why is the red one next to the yellow one?" The four-year-old child understands and explains that the order of the counters in the truck is dependent on the itinerary of the truck: if it goes to the blue doll first, the blue counter is first; if it goes to the yellow doll next, the yellow counter will be next to the blue counter; and so on. But, surprisingly, if the child is asked to reconstruct the truck's itinerary, he is incapable of doing so: the mapping is only one way, and he does not understand that the order of the counters in the truck determines the itinerary just as the itinerary determines the order of the counters.

Incomplete though it may be, this semilogic is an important development and a necessary stage which the child has to pass through before he can acquire reversibility. The well-known experiment with the balls of clay can be reformulated in terms of functional dependencies. At first, a dependency is established between actions and their effects. If one rolls the clay (x), it becomes longer (y_1):

$$y_1 = f_1(x).$$

But if one rolls the clay (x), it also becomes thinner (y_2):

$$y_2 = f_2(x).$$

Both dependencies may be thought to covary; that is, if one rolls the clay, it gets longer and thinner. Finally, the child is able to express this covariation between y_1 and y_2 directly, without the necessity of linking it to the action of rolling, itself. A function (f) that is reversible (f^{-1}) is seen to exist between y_1 and y_2, whereby getting longer is exactly compensated for by getting thinner, and vice versa:

$$y_1 = f(y_2) \quad \text{and} \quad y_2 = f^{-1}(y_1).$$

From Concrete Operations to Formal Operations

Around the age of six or seven the semilogic of the preoperational period starts to turn into logic. At first this remains a limited logic—hence the term *concrete* operations—in contrast with the "full" logic of formal operations. However, this term does not mean that the child can think logically only if he can at the same time manipulate objects. Even less does it coincide with the (rather difficult to define) distinction between abstract and concrete. *Concrete,* in the Piagetian sense, means that the child can think in a logically coherent manner about objects that do exist and have real properties and about actions that are possible; he can

perform the mental operations involved both when asked purely verbal questions and when manipulating objects. The latter situation is far preferable to the former, mainly for reasons of clarity, but the actual presence of objects is no intrinsic condition. Nor is the reverse—that is to say, the absence of objects—a condition for formal operations; these may indeed involve the solving of problems dealing only with propositions, but they may, and usually do, apply to quite concrete situations.

Among the many problems that come within the reach of adolescents at the level of formal operations, there is one (designed by B. Inhelder) that, I think, makes this aspect of the distinction quite clear. The experimenter shows the child a collection of metal bars (some made of brass, others of aluminum; some cylindrical, others with a square cross section; all of various lengths) which can be fixed above a board and then weighted at the end so that they will bend. The problem the child has to solve (which is unsolvable until about the age of twelve, i.e., the formal-operational period) is, Which bar bends most? The various questions are posed: "A long, brass, cylindrical one? A short, brass, cylindrical one? A long, aluminum, cylindrical one? A long, alumnium, square one? . . .?" To work out this problem, all the properties except one must be kept constant during the comparisons: for example, the child might compare two brass rods of the same length to see if the round (cylindrical) one bends more than the square one. The problem cannot be solved in any direct way. A direct, concrete solution would necessitate the existence of rods that are not made of anything at all, have no cross section and no length. Such rods do not exist—in fact, cannot exist—and one cannot even have a mental image of them. But by comparing two rods made of the same metal, with the same cross section and of different lengths, one *creates* impossible rods—in this case, rods that have only length. The concrete-operational child can do no such thing; he can manipulate and think about real objects, but he cannot work with hypothetical entities. He will not be able to solve the problem until about the age of twelve, when he attains the formal-operational period.

The stage of concrete operations results in an important change in children's manner of thought. They now possess what Piaget has called the structure of *groupement* (an incomplete but grouplike structure of transformations comprising invariants).

The term *structures* has given rise to many controversies. Questions such as the following are frequently asked in connection with Piaget's structural approach to intelligence: Have the structures any psychological reality? Or are they no more than a psychological artifice? What is the use, if any, of the search for such structures?

Many disciplines reach a point in history where their subject matter

becomes so varied and the types of problems dealt with so large in number that the need is felt for some kind of unification. In mathematics, for instance, the foundations for organizational principles were laid in the beginning of the twentieth century and led to the mathematical theory of sets and mathematical logic. In psychological development, a similar need is at the basis of Piaget's search for structures.

At a certain stage the child becomes capable of dealing with a great variety of problems. Obviously, he is not aware that he is reasoning according to certain well-defined principles, much less that he is using structures. But the way in which he reasons clearly indicates that there is some kind of organisation. What type of organisation and what sort of general mental operations can account for the way a child at a particular stage solves certain problems and yet fails to solve others? It is easy to observe whether a child gives the right answer to a certain problem; it is much more difficult to observe how he goes about solving it. To account for the method of solving (or of failing to solve) a variety of problems, one has to go beyond observation and suppose the existence of an underlying system of operations (a structure).

Concrete operations

The operations that form the concrete "grouping" are of the most general kind (putting objects together into a class, separating a collection into subclasses, ordering elements, ordering events in time, etc.). These operations are transformations that are reversible, either through annulment (as in the case of adding, annulled by subtracting) or through reciprocity (as in the case of relationships: A is the son of B, B is the father of A).

The importance of the concept of group structures goes beyond the realm of mathematics. In the natural sciences one can, in certain cases, postulate the existence of a grouplike structure. When one then considers the effect of certain transformations on a set of object states (elements), it becomes possible to hypothesize the existence and even the nature of some previously unobserved object states.

Groups of transformations possess an identity operation. Similarly, when one deals with actual problem situations where objects are displaced or changed in form, certain modifications have no effect on certain quantitative properties. For example: pouring a liquid into a different glass has no effect on its volume; kneading a substance has no effect on its mass; spreading out counters does not change their total number.

In Piaget's approach, one learns about the structures of thought by studying, for instance, at what age and in what manner children conceptualize the invariance of quantitative properties such as weight and

volume. These are, of course, the famous "conservation experiments." Because of the fact that invariants are always invariants of a system of operations, the acquisition of the conservation concepts is an excellent indicator of the level of intellectual development.

For the concrete-operational period, Piaget distinguishes the basic transformational structures of classes on the one hand and those of relations on the other. *Classification* implies the grouping of objects according to their similarities; *seriation* implies the ordering of objects according to their differences.

The system of operations that accounts for the problems in classification that eight- or nine-year-olds can deal with can be formalized as follows:

If A, B, C, and so on, are classes that are included one in the other and A', B', C', and so on, their complementary classes, the following operations pertain.[1]

1. $A + A' = B$; $B + B' = C$; and so on.
2. $B - A' = A$; $C - B' = B$; and so on.
3. $A + 0 = A$.
4. $A + A = A$; $B + B = B$; and so on.
5. $(A + A') + B = A + (A' + B')$, but $(A + A) - A \neq A + (A - A)$.

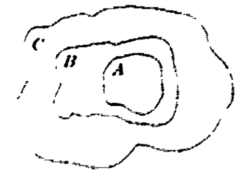
In this system of operations, reversibility is annulment: adding A' to A gives B; subtracting A' from B gives A.

This structure accounts for the success achieved by children at this

1. Editor's Footnote. It is instructive to translate Piaget's language and symbolism into current mathematical language and symbolism. Readers with a mathematics background will be more familiar with the latter. There are certain correspondences between symbols and words which should first be made clear: class $\longleftrightarrow$ set; $+ \longleftrightarrow \cup$; "$'$" corresponds to set difference—that is, A' is not the complement of A but is the set difference relative to a set B; and $A' = B - A$, where "$B - A$" denotes a set consisting of those elements of B that are not elements of A.

Piaget's language

1. $A + A' = B$
   $B + B' = C$
2. $B - A' = A$
   $C - B' = B$
3. $A + 0 = A$
4. $A + A = A$
   $B + B = B$
5. $(A + A') + B' = A + (A' + B')$
   but $(A + A) - A \neq A + (A - A)$



Mathematical language

1. $A \cup (B - A) = B$
   $B \cup (C - B) = C$
2. $B - (B - A) = A$
   $C - (C - B) = B$
3. $A \cup \emptyset = A$
4. $A \cup A = A$
   $B \cup B = B$
5. $(A \cup (B - A)) \cup (C - B) = A \cup ((B - A) \cup (C - B))$
   but $(A \cup A) - A \neq A \cup (A - A)$

level when dealing with several problems in classification: (1) the quantification of class inclusion—they can answer questions on the relative numerical extension of a general class B compared with a subclass A; (2) multiplication of classes—they can find missing elements in double-entry tables; and (3) intersection problems—given, for instance, a collection of pictures of green objects and a collection of pictures of leaves, they can find the green leaves which form the intersection.

Seriation implies reversibility by reciprocity and not by annulment. For instance, in seriating lengths, one has to understand that element E is simultaneously bigger than all the preceding ones and smaller than the succeeding ($E > D$, and, the reciprocal relationship, $D < E$). Moreover, once this problem is clearly understood, a new deductive way of composition becomes possible through the application of a transitivity argument: if $A(\mathrm{R})B$ and $B(\mathrm{R})C$, then $A(\mathrm{R})C$.

The existence of this structure, very similar to that of classification, accounts for the success achieved from seven years onward in problems such as: (1) ordering sticks in an operational manner—that is, first taking the smallest (or biggest) of all, then the smallest (or biggest) of those left, and so on; (2) ordering dolls, walking sticks, and rucksacks of different sizes so that one obtains corresponding seriations; and (3) ordering according to two different properties—for instance, counters of various shades of blue and various sizes to be arranged in a double-entry matrix (e.g., keeping size constant horizontally, colour ordered from pale to dark, and keeping colour constant vertically, size ordered from small to big).

This concrete-operational period stretches from age seven to age twelve and at the same time constitutes a complete elaboration of the types of reasoning made possible by these operations, an application of their power to more and more difficult contents, and a preparation for the much more powerful formal operations.

Formal operations

Since the concrete operations, as I have just said, bear only on reality in the sense that they are applicable to true and observable situations (whether the situation is actually present or not), the novelty of formal operations is that they can bear on hypotheses—that is to say, on statements that are not known, nor supposed to be true at the outset—and on behaviour and properties of objects that cannot be directly observed. In formalized terms, this means that propositional logic becomes possible, admitting implication (if . . . then), disjunction (either or both . . . or), exclusion (either . . . or), incompatibility (or . . . or . . . or neither nor), and so on, between propositions. In terms of groups of transformations, this implies that annulment by inversion and reciprocity become com-

bined and that therefore every transformation is now at the same time the inverse of another and the reciprocal of a third.

Let me give just one example of the possibilities that are now opened up as far as the reasoning of children at this level is concerned. In a situation where an object is shown to move and to stop while a light comes on or goes out, a child of this age is capable of the following reasoning—which can be observed through the experiments he does, since he can manipulate the object. The first hypothesis might be that the light is the cause of the stops ($l \Rightarrow s$). This would mean that $l \, . \, \bar{s} = 0$. But there could still be stops without light ($\bar{l} \, . \, s$ need not be 0).[2] If, on the other hand, the stop causes the light, then there could not be a stop without light ($\bar{l} \, . \, s = 0$), but there could be light without stop ($l \, . \, \bar{s}$ need not be 0). Therefore: (1) if $l \Rightarrow s$, then $l \, . \, \bar{s} = 0$; and (2) if $s \Rightarrow l$, then $\bar{l} \, . \, s = 0$. If, on the one hand, the light occasionally comes on without the object's stopping, then hypothesis (1) is invalidated; if, on the other hand, the object stops occasionally without a light, then hypothesis (2) is invalidated. In Piaget's formalization, these operations constitute a group of four transformations such that $N = RC$, $R = NC$, $C = NR$, and $I = NRC$. Here I represents the *identity* transformation; N represents the *negation* transformation; R represents the *reciprocity* transformation; and C represents the *correlativity* transformation. The group table is shown in figure 1. From the table one sees, for example, that $RC = N$ (from the R-row and C-column intersection), $NC = R$, and $(NR)C = I$. This system unites both inversions and reciprocities, which remained separate in the incomplete grouplike structure of the concrete operations.

	I	N	R	C
I	I	N	R	C
N	N	I	C	R
R	R	C	I	N
C	C	R	N	I

Fig. 1

This very brief sketch of the main stages of cognitive development leaves many aspects untouched. In particular, nothing has been said about what is called the *semiotic function*, the peculiarly human capacity to represent objects and events by something else. This "something else" is not necessarily language, although language is certainly the most important part of the semiotic function; mental images, gestures, symbolic play, and, even before any of these behaviours can be observed or inferred,

2. Editor's Note. Here "." means *and* and "0" means *false*.

imitation in the absence of the model, are all part of this capacity to represent reality (in the largest sense of the word). Evidently, this capacity of re-presenting reality—that is to say, of rendering it present—extends the field of mental action enormously. It liberates the child from the limiting constraints of the here and now: it enables him to recapitulate past events and to anticipate future events. In short, from an organism that reacts and acts in the face of present circumstances (according to the actual situation), the infant becomes an individual who can begin to "know" and to plan.

I have been asked to devote one period to the Genevan language experiments, and a longer discussion on the semiotic function is needed as an introduction to that paper. However, the representation of reality will also be discussed (in the case of mental images and their observable expression, drawings) when I talk about the different types of knowledge. In fact, if this first sketch has given the impression that cognitive development is primarily or even uniquely a development of logic, I hasten to emphasize that this is not the case. There are many different types of operational structurations, and certain types are theoretically distinguished one from the other.

KENNETH LOVELL

Some Aspects of the Growth of the Concept of a Function

The study I am going to describe attempted to measure the extent to which the concept of a function had been mastered. It was carried out at Leeds by A. Orton (1970).

Recent work by the Geneva school (Piaget. Szeminska, and Bang 1968) deals with the growth of understanding of some aspects of the concept of a function. It can be shown that thé thinking of the preschool child may be characterised by a number of one-way mappings or functions, which contain qualitative identities but no real invariants. For Piaget and his colleagues these one-way functions, functions in process of formation, or contributary functions—however we care to call them—represent, as it were, points of departure for the elaboration of what Geneva calls *well-formed functions.* However, it must be pointed out that the experiments that Piaget, Szeminska, and Bang used to study the growth of these well-formed functions were linked with the scheme of proportionality, for only those functions in which laws of variation play a part were considered. A function was considered as a relation between the magnitude of two quantities, the variation in one bringing about the variation in the other in the same proportion.

The present-day mathematical definition of a function is more general than that considered by Piaget. A function from a set X to a set Y is a *relation* such that if $x \in X$, then there exists a unique $y \in Y$ that corresponds to it. Nevertheless, an understanding of a function as now defined in mathematics is dependent on Piaget's stage of formal operational thought and the elaboration of second-order operations. The pupil must be able to handle ratios between ordered values of variables and also the

concepts of continuity and limit, all of which are sometimes involved in an understanding of functionality.

The only previous study of the development of the concept of a function in mathematics in pupils of high school age was carried out by Thomas (1969). Indeed, this was the pioneer study. Thomas used a group test on functions with 201 seventh- and eighth-grade pupils with an average age of thirteen years. The mean I.Q. of those for whom an I.Q. was available was 125, so he appears to have been studying pupils who were well above average in ability. The next phase of his study involved the selection of twenty subjects for individual testing. This was effected by random selection from subsets, these subsets being defined by group-test response patterns, by age, and by sex. The responses to sixteen tasks suggested four stages in the growth of the idea of a function.

The Leeds Study

In the Leeds study all the pupils involved had a background knowledge of sets, operations on sets, ordered pairs used for a variety of purposes, graphical representation of ordered pairs, and elementary directed numbers introduced through the manipulation of vectors defined as directed line segments and expressed as ordered pairs. Other work often demanded a revision and extension of the concept of a function—for example, the study of geometrical transformations in two dimensions as a mapping of one set of points to another, or the study of differentiation as a limit connected with the ratio of intervals of the number line mapped onto itself. The point is that the concepts of relations and functions were present in mathematics from the moment they were introduced, and no pupil could avoid meeting them in each year of school mathematics after their first introduction.

It should perhaps be said that for the purpose of this study, and generally for British school mathematics, *function* is used in the sense of *single-valued function*. The function $y = f(x)$, defined on X as domain and with a subset of Y as range, gives a mapping of the set X into the set Y such that for each $x \in X$ there is a unique image $f(x) \in Y$.

The subjects

The subjects were all pupils in a mixed, comprehensive secondary school (including eleven- through eighteen-year-olds). Eight boys and eight girls were selected from each of the second through the fifth years, together with eight students in the sixth year. This gave a total of seventy-two subjects whose ages ranged from twelve to seventeen years. The eight pupils in their sixth year were highly select in that all were

studying courses leading to the G.C.E. A-level mathematics examination. There were only two girls in this group. The pupils from the other year groups were chosen from the top four mathematics sets, so that almost all the subjects were from the upper half of the ability range. But subjects were chosen to give a spread of mathematical ability within each set, such ability being judged by the students' previous examination results together with the opinion of their mathematics teachers for the year.

The function tasks

The function tasks given are indicated in the Appendix to this paper. The fourteen tasks in part 1 tested a wide range of situations and presented relations in all of the major representations—by diagram, by graph, by ordered pairs, by table, and by equation. Further, in addition to the ability to recognise a function, the formation of the appropriate range from a given rule and domain was considered to be an important part of the tasks. The idea of an inverse was introduced in task 9 and was used thereafter.

It was not until the fourth year that pupils were introduced to the idea of the composition of two functions. Thus, part 2 tasks were given only to pupils in years four, five, and six. In addition, the part 2 tasks used a more advanced notation, the *f*-notation, and used rather harder relations throughout.

Each pupil was interviewed individually, the time required for an interview ranging from 1 to 2½ hours, with an average time of 1¾ hours. The tasks were presented on individual cards, but follow-up and supplementary questions were given orally. There was no time limit for any task. Pupils' responses were tape-recorded and later transcribed.

Function items and scoring procedure

The responses to the subdivisions of the function tasks were regrouped to form items, each item relating to just one aspect of functionality. Thus item 1 brought together all responses to "Is the relation a function?" when the relation was presented as an arrow diagram. The relevant tasks for this item were 1(vi), 2(iii), 2(iv), 2(v), and, to a lesser extent, 6(iv) and 10, used only when additional evidence was required. The subdivisions of the function tasks were regrouped into sixteen scoring items for part 1 and a further six for part 2. The items for part 1 were as given below with the corresponding task numbers in brackets.

1. Does the arrow diagram represent a function? [1(vi); 2(iii), (iv), (v); 6(iv)]

2. Does the rule define a function? [3*a*(iii), 3*b*(iii), 4(ii), (iii)]
3. How must the arrow diagram be altered? [2(iv), (v)]
4. Stating a discrete range. [4(iii)]
5. Stating a continuous range. [3*a*(ii). 3*b*(ii), 4(iii)]
6. Describe the relationship in words. [3*a*(ii), 3*b*(ii), 4(iii)]
7. Finding images and pre-images from graphs. [5(i). (ii) : 7(i), (ii)]
8. Domain and range from a graph. [6(i), (ii)]
9. Convert a graph into an arrow diagram. [6(iii)]
10. Does the graph represent a function? [5(iii), 7(iii)]
11. Ordered pairs. [8(i). (ii) ; 14(v), (vi)]
12. Tabular form. [9*a*(i), (ii) ; 9*b*(i), (ii)]
13. Problem concerning lockers. [11 and 12]
14. Mapping of square to circle. [13]
15. Time/height/weight/speed problems. [14(i), (ii), (iii)]
16. Difference between relation and function. [14(iv), some reference to 14(v), (vi)]

The items for part 2 were the following:

17. Composition of functions defined on discrete domains. [15*b*]
18. Composition of inverse relations defined on discrete domains. [16*a*. *b*]
19. Range for mappings of real numbers. [17*a*]
20. Composite functions with equations. [17*b*]
21. Composition of inverses of functions with equations. [18]
22. Equations for inverses and for composite relations. [17*b*, 18]

Responses to the items were assessed on a five-point scale. In order to define the criteria for the scores for each item, each of the responses was studied and common levels noted. For example, the criteria for the five levels of response to item 15 were the following:

1. Unable to attempt or incorrect attempt
2. A realization of what the situations are about and an attempt to explain in terms of type of relation or arrows, but not up to level 3
3. Correct answers and explanation for task 14(i), but considerable confusion over either 14(ii) or (iii)
4. Correct answers to all parts, but explanations showing some confusion or unnecessary complication
5. All parts answered well

The criteria for the five levels of response in respect to item 20 were:

1. Unable to attempt or incorrect attempt
2. Able to make some correct statements about domain and range, probably in terms of "first set," "middle set," and so on
3. Able to answer (i) correctly with or without further questioning, but unable to complete (ii) even with further questioning
4. Able to answer (i) correctly without further questioning and able to complete (ii) after further questioning
5. Able to answer both parts without further questioning

Other tests

All subjects worked the AH4 test—a test of verbal and nonverbal reasoning. It is often used in Britain as a reasoning or general-ability test. In addition all pupils worked, individually, five tasks involving number sequences and proportionality. The verbal responses of each pupil were tape-recorded as in the other tasks. These tasks were taken, with some amendment, from those used by Lovell and Butterworth (1966). But in the analysis of the results, only the scores on the last two tasks were used, since these are most closely linked with numerical applications of the concept of proportion. Thus task 4 involved the pupil's finding the missing number in the following example and explaining how he obtained it:

8 is related to 6
28 is related to 21
10 is related to 7½
? is related to 9

The scores awarded were:

1. The subject recognises that the answer should be larger.
2. The subject attempts unsuccessfully to apply trial hypotheses other than differencing.
3. The subject attempts differencing.
4. The subject obtains a correct answer with differencing.
5. The subject obtains a correct answer with intuitive use of ¾.
6. The subject can symbolise or otherwise verbalize the ¾ correspondence.

Analysis of the results

We have already said that Thomas suggested four stages in the growth of the idea of a function (1969). But in view of the facts that tasks involving operations on functions were not included in part 1 of the present

study and that one of the purposes of part 2 was a study of the operation of composition, it was necessary to redefine the stages in order to classify responses to the function items. Stages 1 and 2 as proposed by Thomas were found to be almost directly relevant to the present study. Orton's (1970) stage 3 for part 1 corresponds closely with substage 3*a* proposed by Thomas, while Orton's stage 4 for part 1 corresponds to the integration of substages 3*a* and 3*b* of Thomas, with a greater degree of generality being shown in the discussion of the concept. The descriptions of the stages used by Thomas have been kept as far as possible in order that the two studies may be compared. Thus part 1 responses were classified by stages according to the following criteria:

Stage 1. The thinking of the pupil is essentially intuitive or concrete in nature. He can carry out processes associated with the function concept when they are essentially arithmetic in character or when the numbers of one set are assigned to those of another by means of a line graph or table. The pupil interprets a rule such as

$$x \rightarrow 2x + 4$$

as a sequence of operations to be performed on some specific number. But the concept of a function as a special kind of relation has not been mastered, and the extension of representation to new and less familiar forms such as the ordered-pair graph is limited.

Stage 2. Pupils still do not understand the basic criteria necessary for a relation to be a function. But they do show a good grasp of the relational aspects of the concept of function, in the sense that for all forms of representations of a function used pupils can find images, pre-images, and sets of images. Further, they are able to identify the domain as that set of elements that are assigned images, while rules such as "add 15," and

$$x \rightarrow 2x + 4$$

are now thought of as operating on any number of the specified domain.

Stage 3. The basic characteristic of this stage is that subjects can identify relations in several types of representation of functions and not functions and can give adequate criteria for each such discrimination. Subjects have mastered the basic concept of a function. They can identify inverse relations as functions or not functions, but they do not always take care to check the uniqueness of images or to check that they have defined a correct domain for the inverse.

Stage 4. Pupils now display mastery of the basic concept of a function to a greater degree of generality than that of subjects at stage 3. All representations of relations can be classified as functions or not functions, with a precise analysis of the uniqueness criterion. Inverse relations are defined with correct domain, and uniqueness of images is checked.

In most cases, stages 1, 2, 3, and 4 of part 1 correspond to item scores of 2, 3, 4, and 5 respectively, but there were some exceptions. Item 4, on stating a discrete range, produced very few spontaneous correct responses (score 5), and an item score of 4 must be considered to represent a stage 4 response, there being no item score equivalent to a stage 3 response.

For part 2 responses, in which the emphasis of the tasks was on the composition of relations, together with the use of notation (f, f^{-1}, etc.), it was more difficult to define directly comparable stages. However, the Leeds stages are:

Stage A. Success with tasks related to the composition of functions and relations, and of their inverses, is limited to finding images by sequencing assignments in the compositions. Domain and range can only be identified in simple cases and by direct reference to a diagram.

Stage B. Subjects are successful with some of the tasks involving composition, and, in particular, are able to define domain and range in simple cases without being restricted to those members contained in a diagram.

Stage C. Pupils can complete tasks involving composition, with some indication that the processes can be thought of as operations on a set of functions. Subjects are able to identify domain and range, including domain and range for composition of inverse relations, but they have difficulty in checking the uniqueness criterion in inverses.

Stage D. At this stage complete mastery over compositions is exhibited and classification of relations as functions or not functions is consistent. Even in the composition of inverse relations the domain is correctly defined and the uniqueness criterion checked.

In part 2, one item, 19, should be singled out as being more appropriately connected with the stages of part 1, since this item did not involve the composition of relations and functions but involved the consideration of the range of functions defined by equations. However, the definition of the part 2 stages corresponds with the definition of part 1 stages with

respect to concept formation, and thus for convenience this item's responses were classified on A, B, C, and D stages. For all the part 2 items, the scores of 2, 3, 4, and 5 correspond to the stages A, B, C, and D respectively.

Results

Table 1 shows the number of part 1 responses at stages 1–4, with the figures in parentheses indicating the percentage of responses. There were 256 responses for each of the years two to five and 128 responses for year-six subjects. In some instances subjects had not even reached stage 1, so that the figures for a year group do not add up to 256 or 128.

TABLE 1

Distribution of Responses by Year Group and Stage: Part 1

Year Group	*Stage* 1		2		3		4	
Second	65	(25.4)	32	(12 5)	27	(10.5)	64	(25)
Third	39	(15.2)	49	(19.1)	42	(16.4)	107	(41.8)
Fourth	45	(17.6)	38	(14.0)	24	(9.4)	130	(50.8)
Fifth	28	(10.9)	50	(19.5)	40	(15.6)	132	(51.6)
Sixth	3	(2.3)	9	(8.0)	18	(14.0)	94	(73.4)

One single sixth-form pupil accounted for all the responses that were below stage 1 in that year group.

Table 2 shows the number of part 2 responses at stages A–D, with the corresponding percentages in parentheses. There were 96 responses in each of the fourth and fifth years and 48 responses in the sixth year.

TABLE 2

Distribution of Responses by Year Group and Stage: Part 2

Year Group	*Stage* A		B		C		D	
Fourth	37	(38.5)	36	(37.5)	7	(7.3)	7	(7.3)
Fifth	30	(31.3)	30	(31.3)	19	(19.8)	8	(8.3)
Sixth	5	(10.4)	12	(25)	18	(37.5)	13	(27.1)

Table 3 shows the mean scores for the proportion and function items as a percentage of the maximum possible. The maximum score for the function items was 5 and for the proportion items 6, these being the scores awarded for the answers for which the teacher would hope and strive. Table 3 shows that the part 1 items are, on our system of scoring,

TABLE 3

Means: Proportion and Function Items
(Expressed as Percentage Correct)

Year Group	Item Type		
	Proportion	*Function (part 1)*	*Function (part 2)*
Second	44.2	58.6	...
Third	56.7	75.4	...
Fourth	65.8	77.2	53.0
Fifth	65.0	82.0	57.4
Sixth	77.5	89.0	76.4

easier than the proportion items, with the latter being easier than the part 2 items.

Intercorrelations were computed between the scores obtained on the function tasks in part 1, the AH4 test, and the proportionality task scores. The scores on the proportionality tasks correlated highest with the AH4 scores. The intercorrelation matrix was subjected to a principal-components analysis which yielded a general factor accounting for 48 percent of the variance. All the loadings on this component were positive and significant. For example, items 11, 10, 12, 1, and 15 all correlated greater than 0.8 with this component. It appears to reflect a strong intellectual and educational dimension at the centre of the pupils' knowledge and recognition of functions in all the different means of representation; it also correlated highly with the items involving problem situations. Finally the scores obtained by the second- and third-year pupils on part 1 of the functions items, also those obtained by the fourth- and fifth-year pupils, were separately subjected to analysis of variance. There were significant differences due to age, ability (score on the AH4 test), and item in both instances, but in neither analysis was there a second or higher-order interaction.

Discussion Generated by the Leeds Results

There is now some discussion of issues arising out of this study which need investigating further. The following points are made by Orton (1970).

Functions and proportions

It has already been indicated that the basic concept of a function is less related to intelligence than is the concept of proportion. At the same time, many functions defined by simple rules do involve proportion. For example, task 4 involves the proportion

$$y - 5:x = 2:1.$$

In this respect the older definition of a function—which required an equation or a law of variation, or a table of values such as would arise from an equation or law—is more associated with proportion than is the more general definition used currently in mathematics. With the present definition of a function it is possible to study functions and relations without the added complication of proportion. But the modern definition does have the disadvantage that it introduces technical terms and some additional notation. And, of course, when functions are defined in the present, more general way, as a type of relation, sooner or later simple equations that involve proportionality must be introduced. The functions defined by equations which summarize more complicated rules of variation, such as the trigonometric functions, would thus appear to be more difficult than some algebraic functions in which the law is relatively simple. Here is a topic that needs investigating.

Many of the second-year pupils had not grasped the basic idea of a function and did not recognise a function even in simple cases. In this age group, some children wanted to define a function as a relation that produced a pattern, or a combination of patterns, when plotted on an ordered-pair graph. The pupil who tries to use patterns to identify functions is using an incorrect definition of function, but it is one that is more closely connected with the old definition of function than with the new. The type of function which produces a straight-line graph, and so involves a law of proportion, was more readily identified by some of the younger children as being a function simply because they had not learned the basic definition of function, and not because functions involving proportionality per se are more readily identifiable. The confusion of many second-year pupils and their desire for pattern suggests that if the modern definition of function is to be used, the property possessed by some elementary functions—namely, that points on the graph form a particular kind of pattern—should not be mentioned too soon. There is insufficient evidence from this study to suggest good ways of introducing functions that involve proportion, but it would seem inadvisable to study such functions until pupils have considerable experience with the general case outlined by the basic definition. The growth in understanding of that subset of functions involving proportionality is a research area that needs attention at once.

Types of relation

A function can be defined as a one-to-one or many-to-one relation, but a serious disadvantage of this approach to a definition of a function was apparent in the younger age groups. There was considerable confusion

between the meaning of many-to-one and of one-to-many. The clearest indication of a reason for this confusion was provided by those subjects who said that the diagram shown in figure 1 is an arrow diagram for a many-to-one relation because many pre-images are mapped onto one image. However, they said, if the number of arrows associated with each member of the sets is counted, then there is *one* arrow leaving each member of the domain, but many arrows arriving at a single image, hence one-to-many.

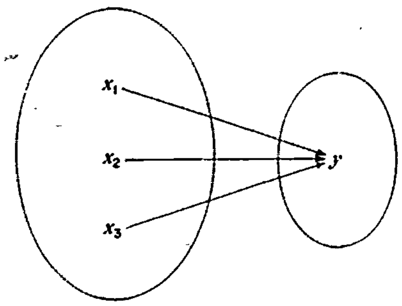


Fig. 1

Our evidence suggests that it may be better to attempt to define a function at first in terms of uniqueness of images of members of the domain—in an arrow diagram only one arrow leaves each member of the domain. The classification of relations into types is also important, but it might be wiser to keep the work on relations apart from the definition of a function at first. One must, of course, admit that in Piagetian terms the failure of pupils to separate the meanings of one-to-many and many-to-one indicates a lack of operative knowing. For in operative knowing, knowing is related to the construction and functioning of the known thing—in our case, a mathematical function. The issue raised in this section needs investigating further.

Graphs of relations

Many of the second- and third-year pupils were unable to interpret the graphical (Cartesian) t[illegible]s with confidence. The principal-components analysis suggested tha[illegible] interpretation of graphs is related to age. There seems to be a strong element of practice or experience involved in the ability to interpret graphs of relations and functions. The graphical tasks involving finding images for given pre-images and vice versa, listing the members of the domain and the range from a graph, and converting the graph of a relation into an arrow diagram or into a set of ordered pairs were difficult for younger pupils. There is a need for further re-

search into the difficulties of pupils in the early stages of learning about graphs of functions.

Relations and functions

The results obtained from item 16, involving the difference between a relation and a function, demand special mention, for many of the responses were unsatisfactory. If functions are approached through a study of sets and relations, there appear to be difficulties. The main difficulty seems to be that many pupils are not clear about the meaning of the technical term *relation* in mathematics. The distinction between the terms *relation* and *relationship* is subtle and not appreciated by younger pupils, and some mathematicians may deny there is any point in making the distinction. It is the relationship or rule defining the relation which children want to take as the relation itself. The term *relation* has more general and varied meanings in everyday life than in mathematics, and children use the word when *relationship* may be a better term mathematically speaking. If the definition of function is to be based on a definition of relation, then it must be made clear that the term *relation* in mathematics does not mean exactly the same as the identical word used in other contexts. Our study aimed to investigate the growth of children's understanding of functionality, and only incidentally has the concept of relation come into it; but it appears that research into children's understanding of relations themselves is necessary.

Continuity

Although the concept of continuity is distinct from that of function the former is, nevertheless, involved in the study of many functions, particularly when the functions are defined by equations. In the principal-components analysis it was found that performance on those items that involved continuous sets or ideas of infinite sets appears to be related more to measured intelligence than to age.

The number line was not used in this study, but research needs to be carried out to establish if with items involving a continuous domain, the number line is a more appropriate pictorial representation than the arrow diagram. It would be worthwhile to establish if the line leads to a greater understanding of mappings defined on the set of real numbers than did the arrow diagram used throughout the present study.

Item 14 also produced interesting results with respect to continuity. In mapping the points around a square onto a circle, and the inverse mapping, some of the older subjects had to think very hard about whether the mapping was one-to-one, and they were clearly considering what happened to points that were close together. This was a difficulty that

younger subjects did not, in general, appreciate, for many of the responses from younger children indicated that they were thinking in terms of discrete sets anyhow. In contrast, the placing of C_1 at the corner of the square, thus making it coincide with a point on the circle, while confusing many younger subjects, did not prove difficult for older ones.

Problem items

Performance on items 13, 14, and 15 was more closely related to intelligence and age than was performance on items that demanded recognition of functions from arrow diagrams, rules, graphs, and ordered pairs. The responses to the question involving school lockers were poorer than expected, although the situation was concrete and familiar to the pupils. Two reasons may be advanced. First, the unused lockers of situation 11(ii) caused confusion, there being a failure to discriminate codomain and range. Second—and more important—the presentation of a many-to-one relation and a one-to-many relation in the last two parts, (iii) and (iv), led to the usual confusion between these types of relation.

Item 15 produced a distinct range of difficulty of questions. The easiest task was the relation that mapped

$$\{\text{age}\} \rightarrow \{\text{height}\},$$

for in this nearly all pupils realised and used the fact that age can only increase. This was not the case in part (iii), which involved time: here pupils did not appreciate that time can only increase. The relation of part (ii),

$$\{\text{height}\} \rightarrow \{\text{weight}\},$$

proved to be a very good vehicle for testing understanding of the concept of a function. It was necessary for subjects to consider all the possibilities that can arise in the

$$\{\text{height}\} \rightarrow \{\text{weight}\}$$

relation. Those subjects who assumed that height and weight varied together were reminded that adults, who normally stay the same height, can fluctuate in weight and that increase in height does not necessarily involve a change in weight. This presented subjects with a variety of possibilities to analyse, some of which implied functionality and others not. Tasks involving familiar situations may pose unexpected difficulties.

Composition

A clear conclusion to be drawn from the responses to tasks involving the composition of functions is the need, among many pupils, for a diagram showing the successive stages in the composition. This argues a lack of understanding of the operational nature of composition. In particular, it was noted that where the diagram accompanying a task de-

picted gf—that is, f first followed by g—responses to gf were relatively good but responses to fg were very poor. An understanding of the composition of functions was arrived at by a few of the abler subjects in the fourth and fifth years.

Equations

Subjects were asked to find equations only in the last two tasks of part 2. The easiest questions were those that involved finding the inverse of a relation for which the equation was given. Much harder was the problem of finding the equation for a composite relation when a diagram was available illustrating the successive assignments. Hardest of all was the problem of finding the equation when there was no diagram available to show successive assignments, irrespective of whether this composition involved inverses or not. Further research is needed to confirm our view that the study of inverses of relations and of the most elementary compositions is appropriate for the more able pupils in the fourth and fifth years.

Conclusions

Some aspects of the concept of a function, introduced in a very concrete manner, can be grasped by pupils in the elementary school who are at Piaget's stage of concrete-operational thought. But it seems that attainment of the early stages of formal-operational thought—Piaget's stage III*a*—is necessary before pupils are able to tackle the tasks indicated in part 1 of this study. The tasks set in part 2 demand a more developed and flexible formal thought, characterised by Piaget's stage III*b*. As in all other content areas, the more the pupil is familiar with the ideas involved in the concept of a function and the more experience he has of handling functions, the more likely—other things being equal—it is that formal thought will be available to him in this content area and hence that an understanding of the concept of a function will develop.

APPENDIX

Part 1

1. Study the arrow diagram shown in figure 2 for a relation that maps

$$\{-3, -2, -1, 0, 1, 2, 3\}$$

into

$$\{0, 1, 2, 3, 4\}.$$

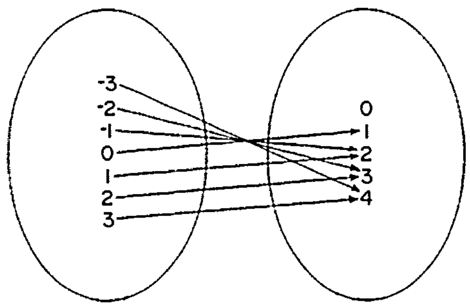


Fig. 2

(i) Write down each image of 2.
(ii) Write down each number that has 2 as its image.
(iii) Write down the domain for this relation.
(iv) Write down the set of images.
(v) Write down the range for this relation.
(vi) Is this relation a function?

2. The arrow diagram in figure 3 shows the relationship "has this number of prime factors" from

{3, 4, 6, 18, 30}

to

{1, 2, 3}.

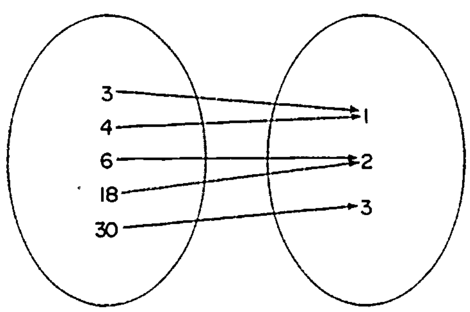


Fig. 3

(i) What is the domain for this relation?
(ii) What is the range for this relation?
(iii) Is this relation a function?
(iv) If the element 12 is added to the first set, how must the arrow diagram be altered?
(v) If, instead, the element 4 is added to the second set, how must the arrow diagram be altered? Is this new relation a function?

3. *a*. The arrow diagram shown in figure 4 is for the relation given by the rule

$$x \rightarrow x + 5.$$

The domain is the set of real numbers, and a few examples are shown.

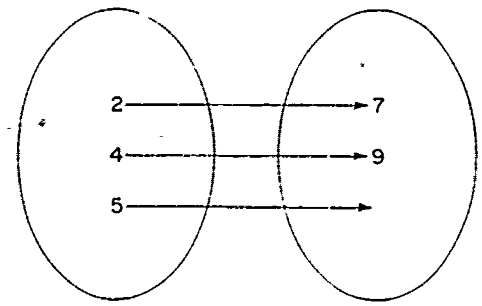


Fig. 4

(i) What is the image of 5?
(ii) What is the range?
(iii) Is the relation a function?
(iv) Describe the relationship in words.

b. The arrow diagram shown in figure 5 is for the relation given by the rule

$$x \rightarrow 3x.$$

The domain is again the set of real numbers, and a few examples are shown.

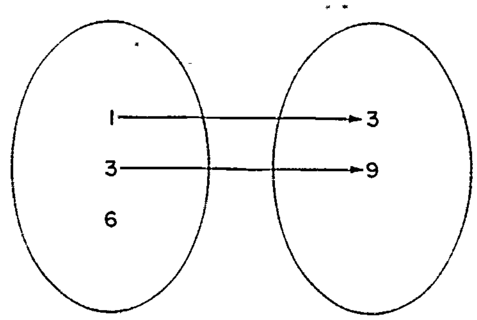


Fig. 5

(i) What is the image of 6?
(ii) What is the range?
(iii) Is the relation a function?
(iv) Describe the relationship in words.

4. A relation is given by the rule

$$x \to 2x + 5.$$

   (i) What is the image of 4?
   (ii) If x is a natural number, that is,
   $$x \in \{1, 2, 3, 4, 5, 6, \ldots\},$$
   what is the range? Is the relation a function?
   (iii) If x is any real number, what is the range? Is the relation a function?
   (iv) Describe the relationship in words.

5. The ordered pairs of a relation are shown on the graph in figure 6.

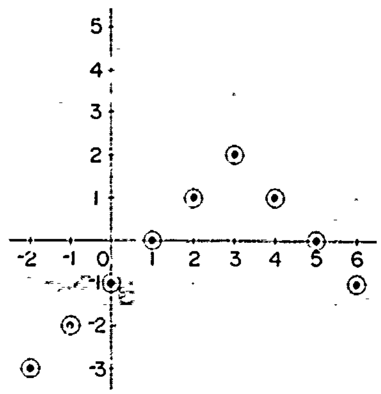


Fig. 6

   (i) Write down each image of 4.
   (ii) Write down each integer that has 1 as its image.
   (iii) Is this relation a function?

6. For the previous relation,
   (i) Write down all the members of the domain.
   (ii) Write down the set of images.
   (iii) Draw the arrow diagram for this relation.
   (iv) Look at your arrow diagram; is this relation a function?

7. Study the graph of ordered pairs for a relation shown in figure 7.
   (i) Write down each image of 3.
   (ii) Write down each integer that has 2 as image.
   (iii) Is this relation a function?

8. For the previous relation
   (i) Write down the set of ordered pairs.
   (ii) Looking at the set of ordered pairs is this relation a function?

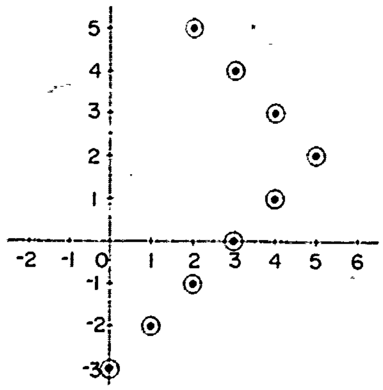


Fig. 7

9. *a.* The table in figure 8 shows the pairs of values for a relation $x \to y$.

x	1	2	3	4	5
y	0	1	1	2	2

Fig. 8

(i) Is the relation a function?
(ii) Is the inverse relation a function?

b. The table in figure 9 shows the pairs of values for another relation, $x \to y$.

x	1	2	3	4	5
y	0	1	2	3	4

Fig. 9

(i) Is the relation a function?
(ii) Is the inverse relation a function?

10. *a.* The arrow diagram in figure 10 shows a relation between two sets of numbers.

(i) Is the relation a function?
(ii) Is the inverse relation a function?

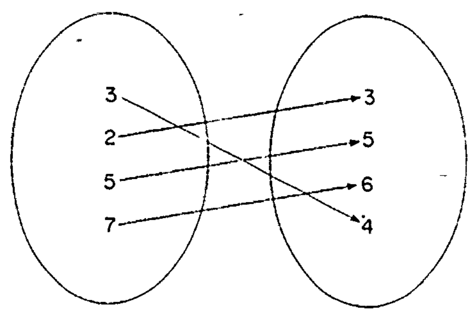


Fig. 10

b. The arrow diagram in figure 11 shows another relation.

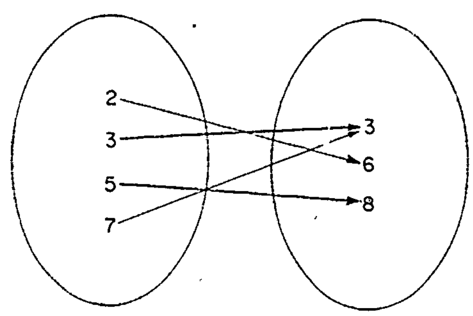


Fig. 11

(i) Is the relation a function?
(ii) Is the inverse relation a function?

11. Class $2X$, whose form room is a laboratory, was given lockers outside the room.

    (i) Every member of the class was given a locker, and there were no lockers left over. Is the relation

    $$(\text{Members of } 2X) \rightarrow (\text{Lockers})$$

    a function?

    (ii) If there were too many lockers but each pupil was only allowed one locker so that some lockers were left unused, is the relation

    $$(\text{Members of } 2X) \rightarrow (\text{Lockers})$$

    a function?

    (iii) If there were too many lockers and some members of the class took an extra locker so that a few of the class had two lockers, is the relation

(Members of $2X$) $\rightarrow$ (Lockers)

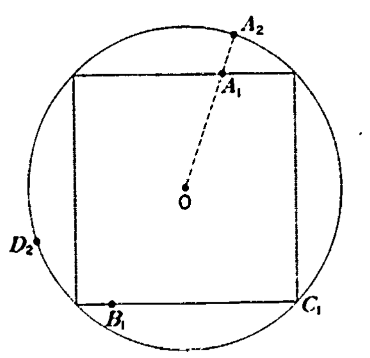
a function?

(iv) If there were not enough lockers to go around and some pupils had to share a locker with someone else in $2X$, is the relation

(Members of $2X$) $\rightarrow$ (Lockers)

a function?

12. Consider the inverse relations of those in question 11 and decide which, if any, are functions.
13. The diagram in figure 12 shows a relation between points on a square and points on the circle that is drawn through the four corners of the square. The points and their images are all drawn so that lines connecting them would pass through the center of the circle, marked 0 on the diagram. Consider the inverse relation of this.



$A_1 \rightarrow A_2$; that is. A_2 is the image of A_1.

Fig. 12

(i) Find the image of D_2 in the inverse relation.

(ii) Is the inverse relation a function?

14. Consider the following:

(i) Is the relation (Your age) $\rightarrow$ (Your height) a function?

(ii) Is the relation (Your height) $\rightarrow$ (Your weight) a function?

(iii) Is the relation (Time) $\rightarrow$ (Speed of a car) a function?

(iv) What is the difference between a relation and a function?

(v) Write down a set of ordered pairs which is a relation but not a function.

PART 2

15. A function f maps a set A onto a set B, and a function g maps set B to set C, as shown in the diagram in figure 13.

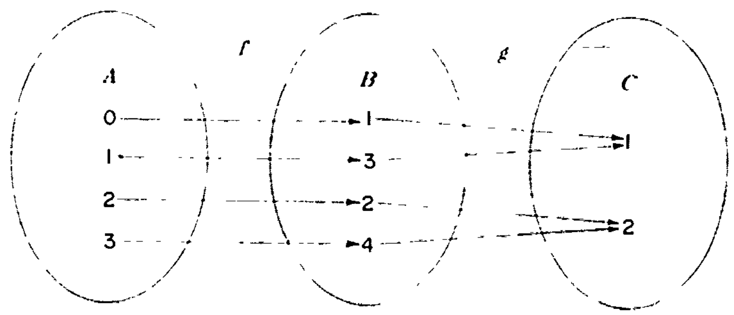


Fig. 13

a. Write down:

(i) $f(1)$

(ii) $g(2)$

(iii) $gf(3)$

(iv) $fg(1)$

b. Answer the following:

(i) What is the domain for gf? What is the range for gf? Is gf a function?

(ii) What is the domain for fg? What is the range for fg? Is fg a function?

16. Draw the arrow diagram for the inverses of f and g in the previous question, using the same sets, and use your diagram to answer *a* and *b*.

a. Write down:

(i) $f^{-1}(3)$

(ii) $g^{-1}(2)$

(iii) $f^{-1}g^{-1}(1)$

(iv) $g^{-1}f^{-1}(2)$

b. Answer the following questions:

(i) What are the domain and range for f^{-1}? Is f^{-1} a function?

(ii) What are the domain and range for g^{-1}? Is g^{-1} a function?

(iii) What are the domain and range for $f^{-1}g^{-1}$? Is $f^{-1}g^{-1}$ a function?

(iv) What are the domain and range for $g^{-1}f^{-1}$? Is $g^{-1}f^{-1}$ a function?

17. *a.* Consider the following:

(i) What is the range if f is the function $x \rightarrow x + 1$ where x is a real number?

(ii) What is the range if g is the function $x \rightarrow x^2$ where x is a real number?

(iii) Complete the diagram shown in figure 14, which shows the two functions together. Four real numbers are used as examples in the domain for f.

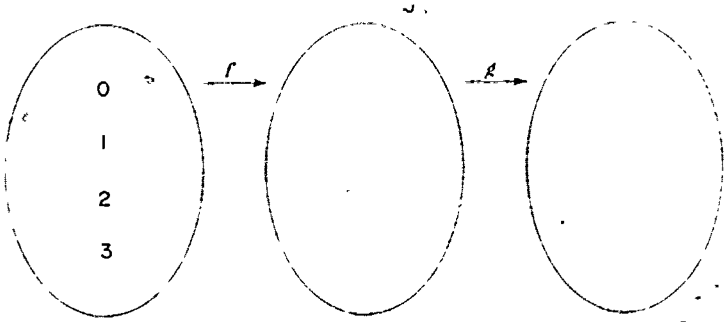


Fig. 14

b. Answer the following questions:

(i) What is the range for gf? Is gf a function? What is gf in the form $x \rightarrow y$?

(ii) What is the range for fg? Is fg a function? What is fg in the form $x \rightarrow y$?

18. Complete the arrow diagram in the previous question for the inverses of f and g.

(i) Is f^{-1} a function? What is f^{-1} in the form $x \rightarrow y$?

(ii) Is g^{-1} a function? What is g^{-1} in the form $x \rightarrow y$?

(iii) Is $f^{-1}g^{-1}$ a function? What is $f^{-1}g^{-1}$ in the form $x \rightarrow y$?

(iv) Is $g^{-1}f^{-1}$ a function? What is $g^{-1}f^{-1}$ in the form $x \rightarrow y$?

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HENRY VAN ENGEN

Epistemology, Research, and Instruction

In 1927 P. W. Bridgeman's book *The Logic of Modern Physics* appeared on the American scene. Bridgeman was concerned about the semantic difficulties physicists were having with such mundane terms as *length* and *time*. These difficulties were brought to the forefront by the Einstein theory of relativity. Bridgeman came to the conclusion that one knows the meaning of a term if it is possible to point to some overt action to which the term refers. In fact, in this book (p. 5) he says: "The concept is synonymous with the corresponding set of operations." Attempts have been made to apply Bridgeman's line of thinking to psychology (Stevens 1935). However, it seems to have had few followers.

At about the same time that Bridgeman's work appeared, Piaget's *The Language and Thought of the Child* was published in English (1926). Piaget, like Bridgeman, places the emphasis on overt actions, or operations.

> An operation is thus the essence of knowledge. For instance, an operation would consist of joining objects in a class to construct a classification. Or an operation would consist of ordering, or putting things in a series. Or an operation would consist of counting, or of measuring. In other words, it is a set of actions modifying the object, and enabling the knower to get at the structures of the transformation. [Piaget 1964, p. 8]

Whether Piaget and Bridgeman ever heard of each other, I do not know. Most certainly as epistemologists they have much in common. The two influenced two entirely different sets of people. In the main, Bridgeman's work was taken up by the philosophers. Piaget's work, on the other hand, has had its major impact on the psychologists.

It is well known to all attending this conference that the ideas of Piaget have stimulated much research on the thought processes of children. Many of these studies have been of interest to those of us interested in mathematics instruction. However, as a teacher of mathematics, I am concerned about the ambiguity and, at times, mathematical error that creeps into the reports of some excellent research. In the words of Parsons (1960), who reviewed Inhelder and Piaget's *The Growth of Logical Thinking*, "One must protest against so much ambiguity and obscurity in the use of logical symbolism" (p. 78). Most certainly, research would be enhanced and communication channels more clearly established if the researcher explicitly stated how certain key words were used and how they fitted into a structure of the subject under consideration. Vague use of such relational terms as *more than* and *shorter than*, as well as terms referring to various aspects of number, too frequently leaves the reader in a quandary. This vagueness also affects the interpretation of the results of the research. Relational terms can be operationally defined and must be so defined for the child in any instructional program. It is the first major task of this paper to clarify and structure some of the ideas that are found in the research literature and in instructional programs. The foundational terms for research and instruction are the same. This is not surprising, since research and instruction have related goals.

Operational Definitions of Certain Relations

The key idea underlying the development of mathematical concepts, even that of conservation of number, is the idea of relation. Relations can, and must, be "operationalized" in order that they can later be applied to number. A specific illustration or two will help clarify the import of such a statement.

The terms *longer than* or *as long as* are frequently thought of as being based on number. It is, of course, possible for an adult to reduce any sentence containing these terms to a number comparison—that is, to restate the problem in terms of a relation between two numbers. However, to think of relations between numbers presupposes the concept of number. For the young child whose concept of number is immature this is not possible.

To define *longer than* operationally, stick A and stick B are laid side by side so that one end of A coincides with an end of B. Then if the second end of A extends beyond that of B, we can say that stick A is longer than stick B. This immediately leads to the study of the relation *longer than*. We discover, by overt actions, that:

1. A is not longer than A.
2. If A is longer than B, then B is not longer than A.
3. If A is longer than B and B longer than C, then A is longer than C.

All these ideas can be established, if it seems desirable, without number and by use of overt actions.

In much the same way, it is possible to define operationally *as long as* and discover that (here "R" stands for *as long as*):

1. ARA, that is, A is as long as A.
2. If ARB, then BRA.
3. If ARB and BRC, then ARC.

The second illustration involves the classical situation in which two rows of objects are placed the one under the other and in one-to-one correspondence. This situation involves the term *as many as*. Here again number is not the fundamental ingredient.

What does the term *as many as* mean? We say there are as many apples in basket A as in basket B if we can match each apple in A with just one apple in B and each apple in B with just one apple in A. These operations define *as many as* in a specific case. The child then learns that we use the term *as many as* whenever there is a matching of elements in set A with elements in set B. This matching must be a one-to-one correspondence.

As many as is also a relation. If "R" represents *as many as* and A, B, and C represent sets, then we write:

1. ARA.
2. If ARB, then BRA.
3. If ARB and BRC, then ARC.

In fact, *as many as* has the same properties that *as long as* has. This is not surprising in view of the way we use these terms. Piaget (1952, p. 55) calls these terms quantitative relationships. From a mathematical point of view, these are relations between sets of objects and not necessarily relations between numbers. Operationally, they are easily defined in terms of actions that do not involve number or quantity. What implication does this have for the classical situation used to test for conservation of number? Suppose we have a row of nickels and another row of candies, with candies and nickels in one-to-one correspondence. We ask a four-year-old child, Are there as many candies as nickels? If the child answers in the affirmative, does this mean that he has some concept of number? Not at all. It may mean that the child has learned how adults use the term *as many as* and observes that the apples and nickels are properly matched. He may have no concept of number.

Now suppose we spread the candies out so that the row of candies

is longer than the row of nickels and ask the same question. Suppose the answer is in the affirmative. Does this mean the child conserves number? Not necessarily. It may mean he conserves the one-to-one correspondence. If he answers in the negative, it could mean that he failed to observe that moving the candies did not destroy the correspondence. The conservation of one-to-one correspondence is the key idea in this situation and not the conservation of number. (In this connection, it is interesting to observe that there is a basic theorem in mathematics which states that if two sets are in one-to-one correspondence, then they are in one-to-one correspondence regardless of how the elements are arranged. It would be interesting to observe when children sense this basic theorem.)

There are a number of other relational terms that can be operationally defined for children and should be so defined for research studies. Among these are *fewer than* and *less than*. It is essential that relational terms be understood (internalized) by the child because he must use such terms to study the order relations for numbers. Failure to recognize these terms as relational terms, and not numerical terms, leads to questionable interpretation of experimental results. For example, Piaget's (1952, pp. 123–57) classical staircase problem (ordering a number of sticks in order of size) is clearly a study of one characteristic of the order relation *bigger than* and is not necessarily a study of ordinal numbers.

But where does number enter into the picture and how? What is number? We now turn our attention to these crucial questions.

What Is Number?

There is more confusion surrounding the concept of number in both instruction and research than surrounding almost any other mathematical concept. There really should be no more mystery surrounding the use of this term than there is surrounding the use of the word *cat*. As a result of this confusion children are confused, teachers are confused, and researchers block lines of communication.

Number involves relations between sets. Two sets A and B are equivalent, written "$A \sim B$," if to each element of A there is a unique element of B and to each element of B there is a unique element of A. Under this condition the following relations hold:

1. $A \sim A$ (reflexive).
2. If $A \sim B$, then $B \sim A$ (symmetric).
3. If $A \sim B$ and $B \sim C$, then $A \sim C$ (transitive).

Now number for the child is simply a noise we all agree to make

whenever we see a set A or any set equivalent to A. Thus, the noise "five" is uttered whenever one is referring to the collection of fingers on one hand. We agree to make this same noise when referring to the collection of pennies equivalent to a nickel. This may startle some people, but it can be put on a sound mathematical basis. This will be done in the next few paragraphs. However, let's look at an analogue. What is a book?

A book is, at times, a collection of objects having the set of properties $\{P_1, P_2, P_3, \ldots, P_n\}$. The symbol *book* then denotes the set of objects having the properties $P_1, P_2, P_3, \ldots, P_n$. The sentence "The book is the foundation of American education" illustrates how the word *book* can be used to denote a class of objects.

On the other hand, *book* also denotes a member of a class. The sentence "The book you see on my desk is not mine" illustrates how, at times, *book* denotes a member of a class of objects.

A child first learns the member-of-a-class meaning of *book* and later the class meaning. In this sense, the child learns to make the noise "book" whenever presented with an object having the necessary properties.

In much the same way, a number, such as four, is a class of objects, namely sets, having certain properties. From the point of view of mathematics, "relations between cardinal numbers are merely a more convenient way to express relations between sets" (Hausdorff 1957, p. 29). That this is a natural way to think about cardinal numbers is brought out by the "empty hat" approach to cardinals. Taking this approach, we define zero to be the empty set.

DEFINITION: $0 = \{ \quad \}$.

Then we set up a means to get a successor to zero. In effect, it is simply "adding one more element" to each set.

DEFINITION: *The set $A \cup \{A\}$ is the successor of the set A.*

It is now easy to write down all the cardinal numbers in terms of set relations:

$$
\begin{aligned}
1 &= 0 \cup \{0\} = \{0\} \\
2 &= 1 \cup \{1\} = \{0, 1\} \\
3 &= 2 \cup \{2\} = \{0, 1, 2\} \\
4 &= 3 \cup \{3\} = \{0, 1, 2, 3\} \\
&\;\;\ldots \\
N + 1 &= N \cup \{N\} = \{0, 1, 2, \ldots, N\}
\end{aligned}
$$

From the above standpoint, the cardinal numbers are only sets of a particular kind. To establish the cardinality of a set K, we find a set

Z in our table of standard sets that is equivalent to K. In actual practice the counting set

$$\{1, 2, 3, 4, \ldots, N\}$$

is used to establish cardinality. The important point to remember is that "four" is a particular set. Then "four" is transferred to all sets that are equivalent to it and called the number of the set. Thus the set of legs of a horse is equivalent to the set {0, 1, 2, 3}, whose name is "4." This enables us to say that a horse has four legs.

As has been observed, in actual practice we order the cardinal numbers and form an ordered set which we call the counting set. To say that a particular number is a set of a particular kind and that the symbol for that set is the symbol for the last element in the counting set is not at all mysterious. Certainly it is no more mysterious than naming a dog. The child learns *dog* as the word applies to a particular dog, maybe a police dog. He then learns to apply it to collies, poodles, St. Bernards, and so on. In fact, he applies it to any member of a whole assemblage of animals. In the same way, a child learns to apply *five* to the set of fingers on one hand. He then learns that *five* is applied to any discrete set of objects that are equivalent to his set of fingers. *Five* is applied to a whole assemblage of sets and we say, "The number in the set is five." All we mean is that the elements of the set can be placed in one-to-one correspondence with the set $\{1, 2, 3, 4, 5\}$.

To complete the mathematical foundation, it would be necessary to establish a system of names, an ordering relation, definitions of addition and multiplication, and so forth. We do not have time to do this. However, a few paragraphs must be devoted to ordinal numbers. It would not be profitable to give a mathematical foundation for ordinals. Hence, I shall sketch the basic idea of ordinal numbers.

Ordinal Numbers

For cardinal numbers the basic idea is that of *equivalence*. For ordinal numbers the basic idea is that of *similarity*. An ordinal number, like a cardinal number, is merely an abbreviated way of talking about sets.

> Two ordered sets M and N are similar when the elements of M and N can be placed in one-to-one correspondence in such a manner that if for any two elements of M, m_i and m_j, the relation $m_i \mathrm{R} m_j$ holds, then for the corresponding elements of N the relation $n_i \mathrm{R} n_j$ holds.

On the basis of similarity, ordered sets can be classified and assigned an ordinal-type. If the set is such that it and all its subsets have a first element, then we speak of an ordinal number.

Finite sets that are equivalent are also similiar. Hence, all sets having the same finite cardinal number also have the same ordinal-type or ordinal number. For practical reasons, we use the same symbol for the ordinal number that we use for the cardinal number when the sets are finite. Herein lies a vast sea of confusion. Both in research and in instruction, the terms *ordinal* and *cardinal* are freely used and almost never defined operationally for the child and in terms of mathematical structure for the researcher.

In the physical world there are events that have a natural order from the point of view of time, place, or arrangement. We could use the alphabet and speak of the *b*th individual to enter a room. This is not usually the case. Instead, we use the standard counting set, which is a well-ordered set, and assign the elements of this set to the individuals entering the room in a prescribed way. The important feature to note here is that "4" is assigned to one and only one object. This is in contrast to the use in the cardinal-number sense where the four is assigned to a whole set. Since the counting set is ordered, we use this ordered set as a communication instrument to order sets of objects. This is the "practical use" of the cardinal symbols to express order.

The development of the cardinals, sketched in this paper, is simple and straightforward. There is no mystery attached. It is possible to define it operationally for the child—a highly important feature. Furthermore, it is not a conglomeration of ideas. In this respect one must disagree with Piaget when he says:

> The whole number is neither a simple system of class inclusions, nor a simple seriation, but an indissociable synthesis of inclusion and seriation. The synthesis derives from the abstraction of qualities and from the fact that these two systems (classification and seriation), which are distinct when their qualities are conserved, become fused as soon as their qualities are abstracted. [Piaget 1967, p. 83]

The difficulty with this conception of number is that it does not distinguish between the elements of a set and the relation that exists between two or more elements of the set. The study of the order of whole numbers is the study of a relation that exists between two numbers and has the usual properties of an order relation. It is one of the many types of relations existing in mathematics. One should not say that these relations are an integral part of the concept of a number, such as six. Assigning the property of a set of objects to individual objects is a common error. This confusion can only block communication, obfuscate essential issues, and delay obtaining reliable answers to research problems.

In much of the reading of psychological foundations of number, one

eries in vain for a precise statement of the experimenter's concept of the ideas under investigation. For example:

> Number is at the same time a class and an asymmetrical relation, the units of which it is composed being simultaneously added because they are equivalent, and seriated because they are different from one another. [Piaget 1952, p. 184]

> Additive and multiplicative operations are already implied in number as such, since number is an additive union of units, and one-to-one correspondence between two sets entails multiplication. [Piaget 1952, p. 161]

These statements and others like them, make the reader demand an explicit statement of the author's conception of number—cardinal and ordinal—and the structure in which number is embedded. This is essential, because the experimenter's interpretations of the child's reactions to mathematical ideas will depend on his, the experimenter's, conception of the mathematical object involved. Hence, in research it is more important to define the object being investigated than to take care of all the statistical niceties.

Because of this failure to state explicitly what ideas are involved, there is reason to question Piaget's conclusions about the interdependence of cardinal and ordinal numbers. Mathematically they are like two parallel roads that are not far apart. One can step from one to the other without much difficulty, but they are different roads and can be traveled independently. Piaget's results are, possibly, only the results of a culture (and schools) that does not make any distinction between the cardinal and the ordinal number. Operationally there is a difference, as will be seen in a later section.

The difficulty with these ideas is nicely illustrated by the title chapters of an excellent little book written for "those interested in children" by Lovell (1961). The chapter titles include "The Concept of Substance," "The Concept of Weight," "The Concept of Time," and similar titles. But on the subject of number the chapter titles are these: "Some Approaches to Number Concepts I" and "Some Approaches to Number Concepts II." Number is a difficult topic!

We now turn our attention to operational definitions of cardinal and ordinal number.

Implications for Teaching and Research

Up to this time, the emphasis has been placed on relations and number. But what do we do to distinguish between cardinal and ordinal number

for the child when testing children and for the purpose of research? We must remember that number is a brief way to talk about certain sets. Instead of saying that the number of fingers on my one hand is $\{0, 1, 2, 3, 4\}$, we say that 5 is the number of fingers on my hand. Here, since $5 = \{0, 1, 2, 3, 4\}$, it makes no difference which response we give. The need for brevity forces us to use "5," however.

When teaching a child how to use these standard sets, we have a choice. We can teach him to associate standard sets (number) with various appropriate sets of objects and then order them, or we can teach the child the standard sets in order. For the purposes of this paper, we choose the latter.

The child must learn to associate *one* with some set composed of a single element and all sets equivalent to it. In the same way he must associate *two* with every set containing a pair of elements; *three* with every set containing a triple, and so forth. We can then order these sets on the basis of "one more" and learn counting.

□ one; □ □ two; □ □ □ three; □ □ □ □ four; etc.
* one; * * two; * * * three; * * * * four; etc.
δ one; δ δ two; δ δ δ three; δ δ δ δ four; etc.

The child should not learn to count as he does on *Sesame Street* by setting up a one-to-one correspondence between the standard sets and the objects. For example, the following figure illustrates this confusing procedure for counting five objects:

□	□	□	□	□
one	two	three	four	five

In the initial stages, the count of five objects should proceed as illustrated below:

□	□□	□□□	□□□□
one	two	three	four

□□□□□
five

The succession of diagrams is supposed to illustrate that a child should set down one block and say "one"; put another block beside the first and say "two"; etc. In this way the child learns that the cardinal number is associated with the whole set and not with an element of the set.

If one wishes to teach ordinal number, one proceeds as in the first

illustration. Here each symbol is associated with one and only one object which indicates the order in which the set has been arranged.

We now have the necessary information to develop further the discussion on conservation of one-to-one correspondence and conservation of number.

Conservation of Number

Suppose that the child has responded negatively to the question "Are there as many red disks in the bottom row as there are black disks in the top row?" (Assume the same number of disks in each row.) What might the situation be? It might be that the child does not know how adults use the relational term *as many as*. That is, he does not know that we are asking him to test for one-to-one correspondence between the two sets of disks. On the other hand, an affirmative reply does not necessarily mean that the child knows there are the same number of disks in each row. He may have only observed the correspondence.

After spreading out one row, the experimenter repeats the question. What conclusions can be safely drawn? If the child has responded affirmatively in this instance, it may well be that the response is based on the perception of the conservation of one-to-one correspondence, not on number. On the other hand, a negative response does not necessarily indicate failure to conserve number. It may indicate failure to conserve one-to-one correspondence.

But what is conservation of number? An activity that would more nearly indicate conservation of number can be described as follows: Suppose a child freely associates *ten* with a set of ten objects. In other words, he knows what is meant by "the number of disks in this pile." Then the experimenter reshapes the pile of disks, splits it into two or three piles or in any way rearranges the disks and asks the question "Now how many disks are on the table?" If the child freely, without recounting, responds "Ten." then it would seem that we could say that this child conserves number.

The ability to recognize that the number remains invariant with the arrangement of the objects is of utmost importance for the understanding of addition and multiplication. Most certainly a child must comprehend that a set of three objects joined to a set of two objects is the same as a set of five objects. That is, number is invariant under this transformation. When this stage is attained, the child can comprehend that "3 + 2" and "5" are really two names for the same number and say, "3 + 2 = 5."

Now let us turn our attention to a restricted list of studies that were done in the Piagetian spirit. They are in a sense side issues insofar as

the main-line theory is concerned, but they are important since they cast light on classroom conditions.

The Wisconsin Studies

Outstanding progress has been made in developing a framework of genetic epistemology based on Piaget's work. For this the teachers of arithmetic should be eternally grateful. Similar progress must be made on classroom applications of Piagetian theory.

One of the early studies in the Wisconsin series was carried out by Zweng (1963). She studied second-grade public school children. Using an operational definition of division, and working with children who had not been introduced to division in the school program, she studied their reactions to the two types of division situations, commonly called partitive division and measurement division. She obtained scores on a set of one-group tasks. A one-group measurement task involved eight pencils to be separated into sets of two pencils each. A two-group measurement task situation involved eight pencils to be placed in boxes with two pencils in each box. Mathematically, and operationally, there is no difference between these two situations; yet the presence of the boxes seemed to be a distraction. Zweng found significant differences between the mean performance of children in one-group situations and that of children in two-group situations. These differences were in favor of the one-group situations. The implications for classroom instruction are obvious.

Van Engen and Steffe (1966) studied the performance of first-grade public school children. These children had studied arithmetic for approximately one school year. A study by Feigenbaum (1963) had suggested that the number of objects in a collection may affect the child's ability to ignore his perception. In reality, the Van Engen–Steffe study was a study of number conservation, although it was not perceived as such at the time.

One hundred first-grade children (fifty boys and fifty girls) were randomly selected and given four tasks. The central idea involved recognizing a number of objects, first as two discrete sets; then, as the experimenter pushed the two sets together so as perceptually to form one set, the children were asked to indicate whether there were the same number of objects present. Initially the experimenters thought that all children would recognize the invariance of the number of objects for "small" sets like two and three, but for larger sets they might not have made the obvious, to an adult, generalization.

Four tasks were formulated. In task 1 the child was confronted with

two sets of candies, a set of two candies and a set of three candies. The child was asked: "If I let you take these candies for your friends, would you take the two piles of candy or the one pile [here the experimenter put the candies into one pile] after I put them together, or does it make any difference? . . . Why?" Task 2 was similar but involved a set of four candies and a set of five. Task 3 involved ten candies and fifteen. Task 4 involved twenty-five candies and twenty-five.

Since the children knew some basic addition facts, they were tested on the combinations they were likely to know, namely, 2 + 3 and 4 + 5. All but one of the children gave the correct response to 2 + 3 on a paper-and-pencil test. All but six knew that 4 + 5 = 9.

Satisfactory responses to the question for each task were categorized under the headings "Same Number," "Same Candy," "Just Know," "No Reason," and the inevitable "Others." A similar classification was made of the unsatisfactory responses. In table 1, the responses to each task are classified.

TABLE 1

Frequencies: Correct Responses by Task and Total Score. $N = 100$

	Task				*Total Score*				
	1	*2*	*3*	*4*	*0*	*1*	*2*	*3*	*4*
Frequency	54	45	45	42	26	27	10	9	28

One does not need to know all about analysis of variance to see the implications of this array of data. All but one of the one hundred children could respond correctly to 2 + 3 on a paper-and-pencil test, yet only fifty-four knew that the number of candies remained invariant. What have the remaining forty-six children learned about arithmetic? The frequency of total correct scores is also revealing.

The twenty-six children who scored zero on the four tasks are most certainly not in any position to benefit from arithmetic instruction as usually practiced in our elementary schools. The situation is even more serious when one considers that the experimenters were actually giving the children an operational definition of addition, although this was not brought to the child's attention. Most certainly the child should not study addition if he does not know that the number of objects is invariant under such transformations as made in this study. A child must be able to conserve number in order to associate meaningfully "2 + 3" and "5" with the union of a set of two objects with a set of three objects.

In 1966, Steffe studied the performance of children on a test covering addition problems. On the basis of a numerousness test, the children

were classified in four levels of conservation of numerousness and, by means of an intelligence test, three I.Q. levels. The problem-solving tasks were classified as to:

1. Physical aids with an explicit transformation
   (*Example:* Four jacks in a pile and three more jacks put with them)
2. Physical aids without a transformation
   (*Example:* Four cars in one parking lot and three cars in another lot)
3. Pictorial aids with a transformation
   (*Example:* Like first example above, but pictorial aids present only)
4. Pictorial aids without a transformation
5. No aids present: verbal description of a transformation
6. No aids present: verbal description of a situation without a transformation

The central question involved the difference, if any, in performance of these twelve groups of first graders randomly selected from some 2,100 first-grade public school children. The principal findings of this study are the following:

1. The problems with no transformations were significantly more difficult than those of all other types, and the problems with physical aids with a transformation were significantly easier than those without a transformation and verbal problems with a transformation.
2. The children in the lowest numerousness level and I.Q. group scored significantly lower than all other groups with exception of four groups in the lower brackets of the 3-by-4 table.
3. The children in the top three levels of conservation performed significantly better than the children in the lower level.
4. The problems with no accompanying aids were significantly more difficult than those with aids.
5. Problems that involved a described transformation were significantly easier than problems without a described transformation.
6. The mean performance in the low I.Q. group was significantly lower than that of the other I.Q. groups.

LeBlanc (1968) studied the performance of children on subtraction problems, using the same population Steffe used, the same classification

of problems, and the same statistical design. Two tables taken from LeBlanc's study are of interest. The possible total score is 3.

TABLE 2

Mean Scores
(I.Q. by Aids by Transformational Type)

I.Q.	*Transformation*			*No Transformation*		
	Physical	*Pictorial*	*No Aids*	*Physical*	*Pictorial*	*No Aids*
1	2.73	2.59	2.21	2.41	2.46	1.71
2	2.57	2.55	2.07	2.16	2.25	1.55
3	2.57	2.50	1.84	1.77	2.09	1.07

TABLE 3

Mean Scores
(Conservation Level by Aids by Transformational Type)

Level	*Transformation*			*No Transformation*		
	Physical	*Pictorial*	*No Aids*	*Physical*	*Pictorial*	*No Aids*
1	2.91	2.94	2.70	2.36	2.85	1.91
2	2.73	2.82	2.06	2.24	2.58	1.73
3	2.61	2.49	1.88	2.36	2.09	1.39
4	2.42	1.94	1.55	1.49	1.55	0.73

The results of LeBlanc's study agree substantially with those of Steffe. I quote from LeBlanc's (1968) study:

> The most significant outcome of this study is the relationship of conservation of numerousness as measured by the pretest to children's performance on a problem-solving test. Although all children received training based on the same curriculum, the performances of the children, categorized into four levels of conservation of numerousness, were significantly different. The children who did well on the conservation test, did well on the problem solving test. Likewise, the children who did poorly on the conservation test did poorly on the problem solving test. [Pp. 154–55]

As related to whether conservation or I.Q. was the stronger indicator of success in problem solving, LeBlanc says:

> With one exception, it was found that, in examining the means of the twelve groups, the conservation pretest related better to problem solving success than I.Q. did. Thus, all the mean performances of the children in the three I.Q. groups of Level 1 were higher than any of the I.Q. groups of Level 2. . . . Thus, the performance on the pretest of conservation of numerousness was a stronger predictor of success on the problem solving test than the group I.Q. test was. However, the two tests . . . taken together

were found to be a slightly better predictor of success in problem solving than either test used separately. [P. 156]

These two studies supply some pretty good evidence that a sizable percentage of our first-grade children are studying arithmetic under adverse learning conditions. This is not only true of the low I.Q. nonconserving group; it is also true of the high I.Q. nonconserving group. It would seem that for these children, the schools should center their attention on activities that might enhance conservation rather than on our traditional arithmetic curriculum. Furthermore, the implications of these studies for the kindergarten program are quite clear.

As a result of the Steffe and LeBlanc studies, the question arose as to whether specifically designed experiences for use in the classroom would affect the ability of kindergarten and first-grade children to conserve numerousness. Harper and Steffe (1968) carried out a study to measure the effect of a sequence of twelve lessons carried out over a period of twelve weeks. The investigators were well aware of the previous work that had been done by such researchers as Churchill, Dodwell, Elkind, and others. This study differed from previous studies in that the test had previously been used in the studies of LeBlanc and Steffe and it was carried out under classroom conditions by a classroom teacher.

Two pretests were administered to experimental and control groups in kindergarten and first-grade classes—the Lorge-Thorndike intelligence test and a test of conservation. One posttest was administered, the test on conservation. Analysis of covariance was used at each grade level where the covariates were the scores from the two pretests and the dependent measure was the score obtained from the test of numerousness. Significant differences were observed between adjusted means of the experimental and control groups at the kindergarten . vel in favor of the experimental group, even though both groups had gained.

Skypeck (1966) studied the relationship of socioeconomic status to the development of conservation. She used three of the conservation tasks in standardized interview-interrogation procedures suggested by Dodwell. The sample for the study was drawn from children in the five-to-eight age bracket found in a large southern city.

Statistically significant differences were found to exist between the scores on the conservation tasks for low socioeconomic groups and average and high socioeconomic g   in favor of the latter groups. The relationship of race to Piagetian development stages was not significant. The findings of this study support the hypothesis that children from low socioeconomic urban environments suffer retardation in cognitive structures related to the concept of cardinal number.

Boe (1966) undertook to investigate the ability of secondary school pupils to section solids. She randomly selected seventy-two pupils from the schools located in a large Wisconsin city. Twenty-four pupils were randomly selected from each of three grades—grades 8, 10, and 12. The subjects were stratified according to sex, grade, and ability. The problem originated, in part, as a result of some years of experience as a teacher of secondary school mathematics and a study of Piaget and Inhelder's theory of space representation. The basic problem involved the sectioning of solids, mentioned by Piaget and Inhelder (1963) in *The Child's Conception of Space*. The subjects were asked to represent the section by a drawing and to select the section from a group of drawings presented by the experimenter. As a result of her study, Boe reported:

1. There are significant differences in the responses made by pupils of differing ability.
2. There are significant differences in the responses to the two tasks, namely, selecting a representation of the section and drawing a picture of the section.
3. Piaget and Inhelder report that by the age of twelve years, all geometric sections had been mastered. This study used sixteen sections included in the Piaget-Inhelder study. Only ten of the seventy-two subjects were successful in the sixteen tasks in the two tests. No subject received a perfect score.
4. Piaget and Inhelder claim that the two tasks are equivalent measures of the same ability. This study reports a low correlation (0.55) between the scores on the two tests.
5. There are significant differences among the responses to the various solid figures. Piaget and Inhelder (1963) state, "The child has 'no greater difficulty' with the cylinder, the prism, and the parallelepiped" (p. 175).
6. Age, as measured by grade in school, was not a significant factor. Ability level, however, was significantly different for both methods of response.

Most certainly this study should be replicated and extended. From the teacher's point of view, it has great potential for supplying useful information. Our schools do far too little to develop spatial imagery.

There are educators who contend that the overt actions used to demonstrate ratio and proportion are different from those actions used to demonstrate rational numbers. Mathematically, in one case one has a linear vector space and in the other an ordered field. Steffe and Parr (1968) devised a series of tests, four on a pictorial level and two on a symbolic

level, to study the performance of fourth-, fifth-, and sixth-grade children in two school systems when confronted with ratio situations and rational number situations. They were interested in equal ratio and fraction situations with missing numerators and missing denominators. For example, the child is shown a picture of two squares and six circles and asked, "If there are two squares for every six circles, for one square there would be how many circles?" The following results were common to both school systems:

1. The pictorial test involving ratios with missing denominators was significantly easier than the corresponding fraction-denominator test for the low and middle ability groups in each grade.
2. The missing-numerator pictorial test for fractions was significantly easier than the missing-denominator test for each ability group in each grade.
3. The high-ability children performed significantly better than the low-ability children on each of the four pictorial tests and two symbolic tests.
4. The fifth and sixth graders performed significantly better than fourth graders on all tests.
5. Very low correlations exist between the scores on the symbolic tests and scores on the pictorial tests.
6. The fraction-denominator pictorial test was the most difficult for each ability group in all grades.

Concluding Remarks

For purposes of research and instruction, American schools and universities need a careful analysis of the relationship that exists between a system of overt acts and the fundamental mathematical ideas studied in the elementary schools. There exists enough evidence, even at this time, that concepts arrive out of physical experience. The study of those experiences that enhance the development of mathematical concepts is sorely needed. The studies reviewed in this paper supply some evidence that the models for physical situations that adults take for granted as being obvious are not obvious to the child. There is some evidence that the absence of an expressed transformation is an impediment to arriving at a solution to a problem. Since this is true, what further difficulties exist in learning to cope with situations for which "$x + 2 = 8$" and "$9 = n - 2$" are models? Some fundamental problems in learning mathematics exist in this general area. Answers to these problems should help us to devise strategies for teaching basic number ideas and mathematical operations.

From the point of view of researching these ideas, we need two things: (1) We need an analysis of basic mathematical ideas in terms of those actions to which these ideas may be isomorphic. (2) Researchers in mathematics education must be more explicit in stating the mathematical structure in which their ideas are embedded and the actions they will accept as evidence that children are in possession of these ideas. Most certainly, a researcher's interpretation of a child's reaction to a given task will be colored by the researcher's own concept of the mathematical idea supposedly under investigation. Furthermore, the tasks used to gather data in an experiment will be conditioned by the researcher's concept of the basic idea, the overt acts he accepts as isomorphic to the idea, and his conception of how this idea fits into a given mathematical structure.

What mathematics education needs is a team of researchers attacking common problems. Studies up to the present have, of necessity, been carried out by one or two individuals. The problems to be solved have epistemological, psychological, instructional, and mathematical tones and overtones. One individual cannot cope with all these aspects. The problems are so broad that any attempt to find solutions needs a team of people who have similar interests but different backgrounds. Let us hope that this conference will point up this need and, in some way, lead to a team approach to the important problems facing the American schools.

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HERMINE SINCLAIR

Different Types of Operatory Structures

Having given a brief sketch of what Piaget considers the characteristics of the main stages in intellectual development, I shall now deal with some of the problems that arise from this theory. Granted that children are, of course, totally unaware of these structures and that the operations only formalize what children can do and how they go about doing certain tasks, it still looks as if these operations are mainly logico-algebraic in character. Are they applied in other fields of activity also? Or, to narrow the question, are they applied in such subjects as physics, chemistry, geometry? If so, how?

It seems useful to introduce a distinction on which Piaget has often insisted, that of the two poles of knowledge. The two types of knowledge that exemplify these two poles are called *logical knowledge* and *physical knowledge*. In the case of logic, our knowledge stems mainly from our own actions or operations and their coordination. To take Piaget's example: a child is playing with a number of pebbles; he arranges them first in small groups, say 2, 4, 6; and then rearranges them so that there are 5, 3, 4; from this he discovers that the total number of pebbles does not change. But the properties of the pebbles have little or nothing to do with this knowledge. The same actions could have been performed upon, and the same conclusion drawn from, a collection of sweets, dolls, and so forth. Only one condition pertains to the objects themselves: they should be clearly separate and should stay where one puts them. Water or milk would not be much good!

By contrast, when a child wants to find out something about floating, it is imperative that the properties of the objects used in the experiment be taken into account. In fact, it is the properties that become the main

source of knowledge. Nothing can be found out about floating without observing how solid and liquid interact. This physical type of knowledge is much less "pure" than the former; in fact, observations by themselves lead only to fragmentary, ad hoc knowledge. To be able to induce laws and regularities and to imagine experiments, a logical framework is necessary. As far as actual situations are concerned, either in real life or in the tasks psychologists propose to children, the two aspects are usually both present but in different ratios. Furthermore, one and the same situation can frequently be used to explore knowledge of different kinds. For instance, the well-known concept of conservation of weight is at first only a conservation of an invariant property of objects: logical operations of addition and composition are sufficient to maintain that weight does not change with a change in shape. But the conservation of weight is all the same more difficult than that of simple substance, a global quantitative concept; and with their growing knowledge of the physical world, children begin to see difficulties they had not considered in the conservation-of-weight problem. Apparent regressions may occur, as in the case of children who begin to wonder whether the ball of clay divided into little bits does not weigh less than the whole ball, since the little bits "press" on more of the surface of the scale, as if an intensive property like pressure were equivalent to an extensive property such as weight. They also often think that the weight of an object diminishes when its movement increases. When one asks the same questions for conservation of weight—(1) after putting the two quantities of plasticine on a double scale; (2) after hanging them underneath the scales; and (3) after hanging two balls at the same height on one side and two balls, one underneath the other, on the other—the children's answers will be quite different at certain stages and there may be apparent regressions. The distinction of the two opposite poles of knowledge is therefore to be seen as theoretical and heuristic; in real situations the type of reasoning applied is somewhere in between the two extremes.

Another question comes to mind concerning this distinction. Given the fact that, starting in the sensorimotor period, all knowledge proceeds from action and that all action implies the transposing or transforming of objects (including the subject's own displacements in space), is the distinction valid at all levels, right from the start of the development of intelligence?

According to Piaget, even at the very early stage of sensorimotor intelligence, that is to say around six months of age, it is possible to distinguish the roots of the two types of knowledge. In one, which will later develop into logical operations, it is mainly the action patterns themselves that become coordinated and integrated. This occurs when-

ever the baby has a specific goal and combines two movements to reach it. In the other, coordination also occurs, but the new knowledge comes mainly from the objects themselves, as in the case of the baby who discovers that if a certain object is pushed and starts swinging, it not only moves (he follows the movement with his eyes) but also makes a sound (he listens to the sound). In a research project on symbolic behavior, carried out in collaboration with Irene Lezine and Myra Stambak, we have examples of children from the age of thirteen or fourteen months onward performing actions that can be classified according to the two types: sometimes they push one object with another; they shake the objects; they tap the floor with them; they explore them carefully with their forefinger. In other instances they spend quite some time (that is, a minute or so) aligning the objects, putting one on top of another, and so on. In the one case, they are learning about the properties of the objects themselves—soft, supple, prickly, and so on—or about the effect of one object on another—one can push the spoon right inside the bristles of a brush but not inside a mirror. In the other case, they seem to be introducing some organization into the objects around them—they put a spoon next to a toy broom and a feather duster—and contemplate the patterns that result from their own organizing action. But most of the time the two types of activity seem at that age to be inextricably interwoven. It is only with continuing development that the activities leading to the two types of knowledge can be more easily distinguished.

There is, apart from the theoretical reasons, an educational use to be made of the distinction. In the case of knowledge of the physical type, reality flatly contradicts an incorrect idea; it is possible to show the child that he is wrong. If a six-year-old thinks that a small piece of iron will float "because it's so light," one has only to perform the experiment for him to see that his prediction is not correct. This does not, of course, mean that he will give up his idea; at first the child is incapable of forming a different hypothesis and will regard counterexamples as "funny" cases. But such a demonstration is totally impossible in the case of a logical problem. For example, if a child affirms that there are more apples than *fruit* (items of fruit) in the bowl, making him count first the fruit and then the apples is not going to have any influence at all. If he can already count and finds that there are six items and four apples, that does not prove to him that there is more fruit than apples. Counting, in this case, can be compared to naming; the fact that the last person named is called Peter does not tell anyone anything about the number of people in the room.

However, there seems to be a bit of a paradox here, which I cannot resolve. Since "wrong" ideas in physics seem to be easily demonstrable

and "wrong" ideas in logic very difficult, why do educational curricula always introduce physics so much later than arithmetic? Is it because every day we make use of technological marvels that only a few of us really understand, whereas our daily contact with mathematics is limited to fairly simple problems such as adding and subtracting money or working out how many rolls of wallpaper are necessary to cover a certain surface? Or is it precisely because wrong ideas in physics are so obviously wrong that it seems hopeless to start teaching physics early? When one constructs the syllogism "Johnny has two eyes; Johnny is a boy; therefore all boys have two eyes," one has all one's facts right, but the logic is way off. On the other hand, when one says, "All boys have two noses; Johnny is a boy; therefore Johnny has two noses," the logic is impeccable, but one of the premises is false. The first type of reasoning, wrong as it is, seems somehow less false than the second!

In his epistemological work Piaget seems to indicate a more profound reason. In many cases, it is only after many aspects of a branch of science have been elaborated that science discovers (or at least explicitly formulates) basic concepts that are acquired early in cognitive development. Such is the case with topological structures. Topology became a branch of mathematics well after Euclidian geometry, but we find that the child can handle relationships of closure as opposed to openness of figures, of being adjacent as opposed to nonadjacent, of overlapping, and so on, well before he can deal with the relations between parallel as opposed to intersecting, curves as opposed to straight lines, and so on. Similarly, one-to-one correspondence, which is at the base of set theory (another recently formulated theory), is one of the earliest established concepts in the child. Possibly awareness and explicit formulation of deeply ingrained, basic concepts comes later than that of more complex achievements. However that may be, a similar phenomenon exists as regards physics, where certain of the most difficult notions of modern physics seem to exist in a primitive, intuitive form in children—but then they look completely "wrong." The concept of time seems to provide an example of this. Very young children have an intuition of speed; this is based purely on the eventual "overtakings" or "catchings up." Time, however—that is to say, duration—remains linked to distance covered or to the amount of work done. Until nine years of age there is confusion when comparing two moving objects or persons as regards the time taken and the distance covered; going further usually implies having taken more time. It is very difficult for children of this age to admit that time is something that can be measured independently of what has been accomplished during the time. In fact, they think that a watch does not work in the same way when it is worn by someone who is running as when it is on the wrist of

someone who is walking slowly. For them, there is no common, homogeneous time. The insufficiency of this postulate in physics became clear only when Einstein used the constancy of the velocity of light, as demonstrated in the Michelson-Morley experiment, to derive his principle of relativity. It seems that, once again, children's intuition has something in common with late developments in science.

The findings of developmental psychology indicate a very clear parallel between the child's reasoning in logico-mathematical problems and his way of attacking problems in physics (generally, to start with, in simple mechanics). For instance, even in the preoperational period, when his logic is still a semilogic of one-way mappings, when a child is asked about the respective lengths of a piece of string in the situation pictured in figure 1, he knows very well that if one pulls on the extremity of A (for instance, by hanging a weight on it), A will get longer and B will get shorter; but since he is as yet incapable of quantification, he will not suppose that $\Delta A = \Delta B$. In general, the child thinks that the gain in the length of A is more than the loss in the length of B; after all, A is "where the pull is" (Piaget et al. 1968). To take another example, where the

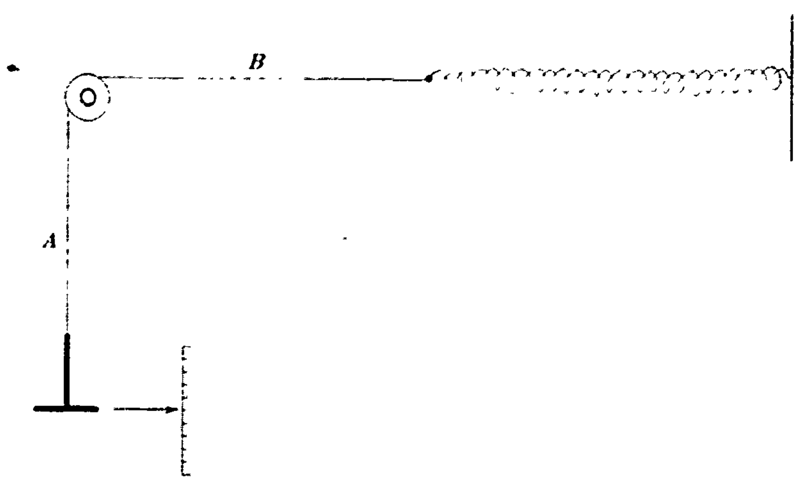


Fig. 1. Spring, string, and pulley task operations

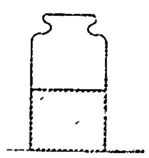
apparent discrepancy between knowledge of physical phenomena and logico-mathematical reasoning is more striking, at the stage of concrete operations, in the following situation (Piaget 1970) a rubber band is marked with a clip at one third of its length and then is stretched out. Children initially think that the stretching occurs only at the ends and then, shortly afterward, at the end of each segment. However, applying the additive compositions of which they are capable at this stage, they

think that the change in length is the same for both parts, despite their initial inequality. They can then observe that their predictions were wrong; but since at this stage they are still incapable of understanding proportion, they will content themselves with the following explanation: "The long bit gets more added to it, the small bit less." The logic of this reasoning is perfectly comprehensible, but it is combined with the idea, as regards the physics of the phenomenon, that the "pull" is not distributed over the whole rubber band but seems to result in "bits" added at the ends. Somehow or other, this idea seems to many people more "wrong" than the corresponding logical concepts, which do not yet enable children to deal with proportionality. Interestingly, as soon as children understand proportionality, they correctly predict the length in this problem and then (as regards the physical aspect) immediately maintain that the "pull" goes through the whole rubber band and that therefore the force is evenly distributed over the whole length.

However, despite this close parallel, a certain time lag in the construction of physics concepts is comprehensible. The logical operations open the way to deduction and do not need any checking with reality, but the physical concepts demand constant checking by experiment. For instance, from very simple premises we can find the following conclusion: if clay retains its total volume when it is split up into little bits (as in the volume-immersion test) and if any shape of clay can be fired to produce a piece of fired clay of the same volume, we can deduce that a piece of fired clay also maintains its volume on fragmentation. In this case, the experimental check is extremely simple, but often such checks are very complex. It may well be that it is this complexity which accounts for the historical time lag between the development of logic and mathematics and that of physics. However, the parallel of the two types of knowledge in child development would suggest that, educationally, the two subjects might be taught in much closer connection than is usually the case.

Where in this scheme of things does one place geometry? Historically, geometry went hand in hand with logic. But in school programs the two became separated. Only recently have educators begun to follow Piaget's practice of amalgamating logic and mathematics (and therefore geometry) in one whole: logico-mathematical knowledge. Once again, experimental psychological findings reveal a remarkable parallel; cognitive development is an indivisible entity, and its laws and the progression of its structures may have different manifestations in different types of problems but nevertheless remain basically the same. In fact, in many experiments a child's physical explanations and logical and geometrical solutions are inextricably linked (as was the case in the experiment on the pulling of the rubber band).

To take an example that has clearer geometrical implications, consider the following experiment (Piaget and Inhelder [a] 1947). The child is shown a squat, angular bottle of ink (see fig. 2*a*). This bottle is put into a cover so that the level of ink is invisible. The child is then asked to draw, in a prepared frame that shows the bottle and the table (see fig. 2*b*). the ink level (i.e., indicate where it is), then to draw its level if the bottle is tilted, put on its side, or turned upside down (all the positions are shown, but with the cover). The first drawings (in the preoperational


(a)

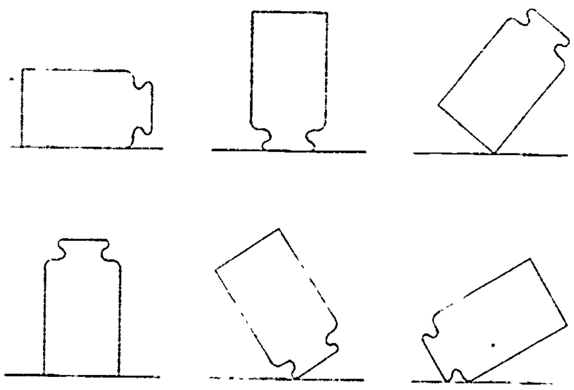

(b)

Fig. 2. (*a*) Bottle with ink. (*b*) Frame positions showing line of the table.

stage) show scribbles that fill the whole bottle (and go beyond as well!). At a second stage, the level of the ink is drawn in all situations as parallel to the bottom of the bottle, and in the upside-down position the ink may be hanging from the top (see fig. 3). In the next stage, certain modifications are introduced, mainly by having the ink go toward the opening and drawing oblique lines that connect the corners (see fig. 4). Finally, of

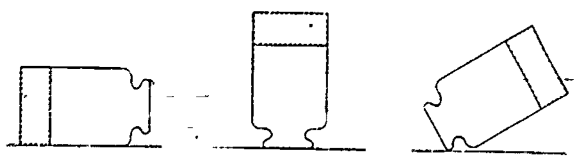


Fig. 3. Examples of drawings at an early second stage

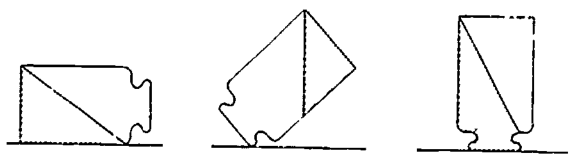
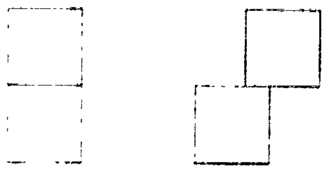


Fig. 4. Examples of drawings at late second stage

course, at about the age of nine or ten, all the drawings are correct. Now these different drawings can be interpreted in a quasi-geometrical way: at first, the only relationship that counts is that the ink be inside (or mainly inside) the bottle. Secondly, the child keeps the intrafigural relationships constant: the level always keeps its relation to the shape of the bottle. It is only at the third stage that extrafigural indications are taken into account and the drawing of the table in the model sketch begins to attract their attention. Finally, a system of coordinates is established, and the children say that the level should always be "straight" or "just like the table." However, their remarks concerning the physical nature of the problem provide a parallel explanation: at first, the ink just stays inside; then, the ink has the tendency to go toward the opening of the bottle (even if, as is obviously the case in this experiment, the opening is sealed). Only in the final stage do they explain that water always goes down as far as it can, that it goes down on all sides, that all bits of water arrange themselves so that the level is horizontal—as long as it is not, bits of water will be higher and roll down toward the lower level until an equilibrium is reached.

Other experiments concern the copying of geometrical figures and, more interestingly, anticipatory images of simple geometrical shapes when they are rotated or translated.

A first example is provided by the following (Piaget and Inhelder [b] 1966). The child is shown two cardboard squares (5 cm by 5 cm), one placed above the other, touching along one side (see fig. 5*a*). He is then asked to copy this situation, and only if he can make a more or less correct copy is he retained for the rest of the experiment. Now the child is asked to imagine something: "How would it look if I pushed the top square a little bit to the right?" (the gesture is sketched but the square is not moved). Once the child has understood this instruction, he is asked to draw (and sometimes to indicate by gestures, but we shall not go into those results) "how it will look." Finally, the experimenter actually pushes the top square, and the child is asked to copy this final situation, figure 5*b*. The subjects were children between four and seven years of age.



(a) (b)

Fig. 5. Task with two squares, 5 cm by 5 cm

Surprisingly, the drawing by anticipation is correct (77 percent success rate) only at the age of seven. Interestingly, even the simple copy of the final state is correct (76 percent) only at five-and-a-half years. Even more interestingly, the deformations introduced in the copy are of exactly the same type as those in the anticipatory drawing, but they appear two years earlier. This fact alone is full of interest and proves once again that it is not sufficient to have the model in front of one's eyes to be able to reproduce its structure. The following types of drawings were observed (here reproduced schematically):

1. The very simplest solution consists in a simple copy of the initial situation, or in a horizontal recombination (see fig. 6).

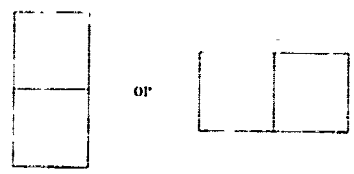


Fig. 6

2. Next (at about the same ages) the two squares are drawn one apart from the other and the top square is moved up instead of sideways (see fig. 7). Fifty-five percent of the drawings of the five-year-olds are of this type.

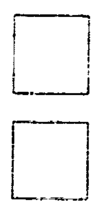
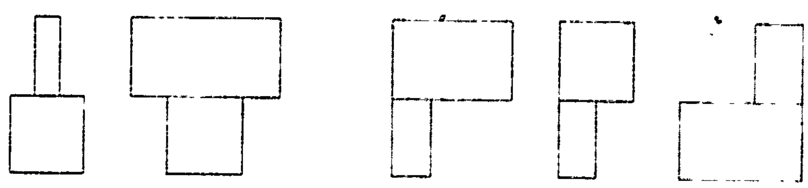


Fig. 7

3. A very interesting series of drawings is found at the next level (see fig. 8). They seem to indicate that the main difficulty resides in the prob-



(a) (b)

Fig. 8

lem of what happens to the right vertical side of the top square when its left vertical side is displaced away from the corresponding left side of the bottom square. The displacement of the right side is represented either as symmetrical to that of the left or vice versa (drawing 8*c*). In other cases, one of the top square's sides is correctly placed relative to the corresponding side of the bottom square; but the top square's other side is drawn in vertical extension of the corresponding bottom square's side (original position), as in drawing 8*b*.

4. Finally, of course, the problem is solved. However, when little vertical strokes are drawn on the squares, on the bottom one to the right of the top side and on the top square to the left of the bottom side, the problem becomes more difficult (see fig. 9). As one of the subjects said: "One of the strokes is near one side and the other one near the other side, so they can never come together." It seems evident that neither remarks such as this nor the deformed drawings can be the result of faulty perception, or of optical illusions. Again, the only way to explain them would seem to be by taking into account the particular character of preoperational thought.

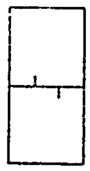


Fig. 9

The solutions indicate that the main problem is ordinal. The relationship between the vertical sides of the top square and those of the bottom square is what the child works on (and fails to solve before the age of seven). The horizontal distance between the two vertical sides is neglected: the shape of the whole (and all children of this age recognize a square when they see one!) is sacrificed. Either the intrafigural relationships are conserved, as in the first stages, or an attempt is made to represent the interfigural relationships, but the two cannot yet be coordinated.

Another characteristic of early "geometric" representation is that correct drawings can be made of the initial and final states of a figure that changes its position in space, but the intermediate stages cannot be represented. In the very simple case of a vertical stick (20 cm long, 2 mm in diameter [Piaget and Inhelder (b) 1966]) that pivots on its base, fixed by a support, children are first asked to draw the stick when "it has fallen a bit" (the movement can be quickly shown by pulling the stick down

through an angle of 20° and 30°). After this drawing, the child is asked to draw the stick in several positions: first when it is upright, then at several points "on the way down," and finally when it "has fallen right down, flat on the wooden support." The experimenter directs the child's attention to the pivot and makes it clear that the stick cannot move away from that point. There is little difficulty, even at the age of four-to-five years, in drawing the two extreme positions. The interesting errors concern the intermediary positions. Figure 10 shows schematized representations of typical errors in indicating the intermediary positions of the stick; the dotted lines represent the (correct) initial and final positions. Solution 3 is particularly popular at four and five years of age. Solutions 4 to 8 are more or less contemporaneous (at five-to-six years of age), and they all show the difficulty of coordinating the movement of the top extremity of the stick (which describes a quarter of a circle) and the successive positions of the whole stick. The curves in solutions 6 and 7 are particularly interesting and represent an incapacity to conciliate the (curved) trajectory of the extremity and the fact that the stick itself does not change its shape.

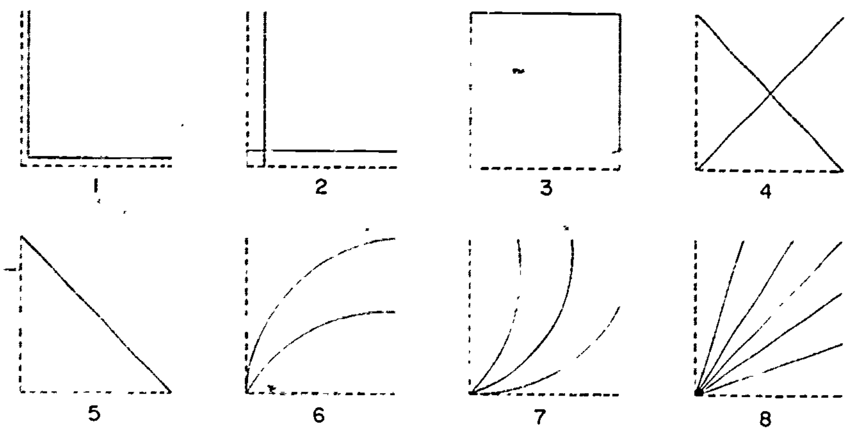


Fig. 10

Should we conclude that geometry follows exactly the same developmental line as that of logical and arithmetical operations? What about the famous mathematical intuition and its supposed reliance on mental images? According to Piaget, this is partly true—especially for the early period—until the first grouplike structure of transformations is firmly established. However, geometry remains a case apart because of the very close correspondence between its operations and their spatial representations (drawing, models). Other experiments have shown how, once the

attachment to the initial and final states is overcome, children around seven or eight years can solve problems with complete and elegant reasoning, whereas young children, though often able to answer correctly, either cannot justify their answers or else cite the wrong reasons. In one experiment the children were presented with a number of drawings (Piaget and Inhelder [b] 1966) such as those in figure 11. They were

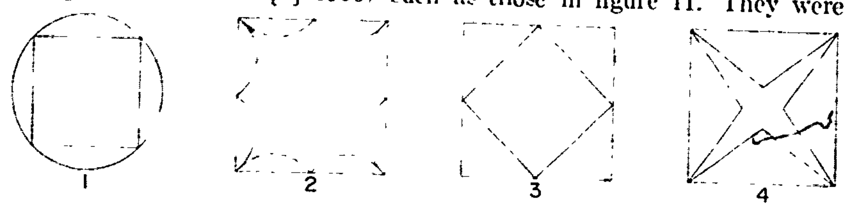


Fig. 11

asked which one of the two lines (two roads, two bits of string) in each drawing was longer, or were they the same? Without giving a detailed analysis of the various stages that characterize children's responses, the following are the main points. At first, from the age of four-to-six years, children think that there is a total correspondence between surface and perimeter, although neither is conserved. Then (seven-to-eight or even up to nine years) the first conservations lead to "wrong" conservation: a change of shape of a surface (for instance, when a square of cardboard is cut into strips which are then glued together into a long rectangle) is no longer thought to change its area, "therefore" the children think that its perimeter cannot have changed either! Conversely, a change in the form of a perimeter (a wire) is thought not to change the surface it delimitates (Lunzer and Bang 1965). In the case of the four figures we have taken in our example, the problem concerning 1 and 3 can be answered correctly by means of simple notions of topology (one surface is included in the other: therefore the outer line is longer, since the inscribed surface is smaller than the including surface). The same type of reasoning, however, leads to wrong answers in cases 2 and 4. In other cases, children below eight years may answer on a numerical basis, this time already reasoning only on the lines and not on the surfaces. In drawing 3, for instance, this may lead to the answer that both lines are the same length: "They each have four bits."

From seven years onward, correct answers are given to situations 2 and 4, and the arguments become based on the possible transformations that would permit a direct comparison: "If you put one of the lines in straight bits like the other, you know it's longer."

It seems clear that in a certain sense geometry constitutes a special case, where representation and mental images are much more adequate

synthesis than in any other domain.

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effective problem solver need "masses of structurally organised knowledge." However, Piaget's position makes clearer the general finding that the level of thought determines the degree of conceptualisation, and the degree of conceptualisation facilitates strategies of thinking. For, in the view of Piaget, to know is to assimilate reality into systems of transformations—to know is to understand how a certain state is brought about.

Reynolds also takes up the question of the construction and testing of hypotheses. He points out that, in Piaget's view, when a pupil begins to think formally, his construction and testing of hypotheses develop together. The results of the study do not fully support this claim. There were occasions when a hypothesis was not recognised as such, when hypotheses were not systematically constructed, and when hypotheses were constructed or recognised but not tested. The questions relating to the potatoes and hard-working scientists and Mr. Smith presented hypothetical situations. The premises in each question were required to be accepted, although no construction of hypotheses was involved, and deductions were to be made only within the framework provided. It might be expected that all pupils at the stage of formal-operational thought would be willing to accept and think within a hypothetical framework, but the results showed that this was not so.

Finally we may note that some of the answers confused data and conclusion. This confusion revealed inadequate notions of the converse of a statement and showed that reversible operations could not always be

The Development of the Concept of Mathematical Proof in Abler Pupils

The concept of proof in mathematics will always be important whatever may be the nature of the curriculum. Professional mathematicians have, of course, analysed the concept of proof—as, for example, in *The World of Mathematics* (Newman 1960). Allendoerfer (1957) is one of a number of people who have given methods of proof together with illustrations at high school level. Again, Fawcett (1938) attempted to teach the nature of deductive proof to high school pupils. But the study of Reynolds (1967), on which this paper is based, is the only one known to me that studied the ways in which pupils of junior and senior high school age develop their concept of proof. The aim of this study was to investigate the development of the understanding of mathematical proof in pupils in British selective (grammar and technical) secondary schools and to see how well this development is explained by the framework provided by Piaget's genetic psychology.

Reynolds starts from the position that a scheme, a generalisable plan of action, or strategy of proof may be characterised as the combination of two processes: the construction of a hypothesis to solve a problem or explain an event, and the construction of a proof or disproof of the hypothesis. In the construction of the hypothesis the rules of logic are generally ~~of~~ little value, for it requires some new combination of the problem solver's knowledge and the data of the problem—there is an attempt to "close a gap." It might be hypothesised, for example, that a certain relationship exists between some variables in a problem, or that a property observed in a finite number of instances can be extended to a

other concepts, what the effects are of continuous teaching from grade one involving small-group work, opportunities of the child to act on reality, much discussion between teacher and pupil and pupil and peers, and a teacher who knows something of the structure of the subject matter. This study only begins to touch this important topic.

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wider class of elements. To the hypothesis and the data of the problem the rules of logic may be applied in an attempt to arrive at a proof or disproof.

Thus when a hypothesis has been constructed its validity must be sought, and for this, methods of proof and disproof are available. Reynolds lists five methods of proof and two of disproof. These are: direct proof; proof by the use of contrapositive; reductio ad absurdum method; proof by enumeration; proof by existence; disproof by contradiction; and disproof by the counter-example method. These will not, however, be discussed here.

Aspects of Proof Considered

Tests were constructed to involve the following aspects of proof: generalisations, symbols, assumptions, and methods of proof. These were chosen for the reasons now briefly indicated.

A generalisation in mathematics is a statement that a property holds for every member of a particular class. In the tests given all such generalisations sprang from a finite number of examples. Thus each generalisation is a hypothesis, and in schemes of proof the making of hypotheses and reasoning from them are important. Assumptions are essential in any argument, and no conclusion is worth more than the assumptions on which it rests. Pupils must be able to recognise their own assumptions and those of others. Again, if pupils are to construct proofs, they must have some awareness of rules of inference and of methods of proof. Attention must be paid to the use of implication and the forms associated with it. Moreover, the reductio ad absurdum method was involved in some questions as it concerns hypotheses and the handling of contradictions. Failure to see a circular argument shows lack of awareness of rules of inference. Finally, facility is needed in the use of symbols; so questions were set to see how pupils viewed mathematical symbols in certain situations, and they were also asked to state a purpose of symbols.

The Experiment

Sample

Subjects were selected from six grammar schools and one technical school. Table 1 shows how the pupils were spread among the age groups.

All the pupils took paper-and-pencil tests. In addition, 80 pupils were interviewed individually. These were 14 girls and 17 boys from the first forms, 11 girls and 6 boys from the third forms, 9 girls and 7 boys from

TABLE 1

Distribution of Pupils by Age

Group	Test 1		Test 2	
	Girls	Boys	Girls	Boys
First form	222	243	186	193
Third form	245	280	240	207
Fifth form	169	199	185	120
Nonmathematical sixth (NM) form	104	22	108	84
Sixth (M) form	29	29	28	67
Total	769	769	747	671

the fifth forms. 5 sixth-form boys who did not take mathematics, and 11 sixth formers who did.

The tests

Test 1 contained 21 questions, and each was answered by every pupil. In test 2 there were 11 questions which were attempted by all pupils; in addition, there were 9 questions set for the first and third forms, and 9 other questions for fifth- and sixth-form pupils. By means of the common questions to every age group it was possible to get some idea of the development with age of the understanding and use of the aspects of proof considered. Eighty minutes was allowed for each of the paper-and-pencil tests, as a pilot study had indicated that this time was ample for most pupils.

The questions in the two tests were placed in six sections for analysis corresponding to the aspects of proof considered, namely, generalisations, symbols, assumptions, and three methods of proof—those using the converse, reductio ad absurdum, and deduction. Examples of questions in each section are now given.

Generalisations. From answers to questions in this section it was hoped to discover how far children accept a generalisation on inadequate evidence, how far they suspend judgment, and what reasons they give for the rejection of a generalisation. An example set for all groups was:

Study the list given:

$2 = 1 + 1$	$6 = 3 + 3$	$10 = 5 + 5$	$14 = 7 + 7$
$4 = 3 + 1$	$8 = 5 + 3$	$12 = 5 + 7$	$16 = 11 + 5$

[and so on, to 16 instances concluding $32 = 3 + 29$]

Do these facts show that every even number can be put as the sum of two prime numbers?

As you know, this question concerns Goldbach's conjecture, whose truth or falsity remains unknown. The sixteen instances given were consecutive and easy to check. Another example was:

> Which of 2^n and $2n + 1$ is larger when $n = 1$?
> Which is larger when $n = 2$?
> Do these values suggest anything about 2^n and $2n + 1$ when n is given other values?

Symbols. As already stated, questions were set to find out how pupils viewed mathematical symbols in certain situations. An example of a question set to all groups was:

> Is $-p$ negative?

Other examples were:

> Is it provable that $2 + 3 = 5$?
> What is the purpose of symbols in mathematics?

It was expected that pupils would indicate how they regarded the symbol "$-$" and any assumption they made about the range of values of p.

Assumptions. A question set to all pupils was:

> What do we mean by a logical statement?

All pupils will have heard the word *logical* used, particularly in support of the way in which a conclusion is reached in an argument. This exercise was an attempt to find out what pupils understand by a *logical statement.* Another question asked was:

> What do we mean by a hypothesis? Give an example if you wish.

Converses. Two questions set to all pupils in the section were:

1. All successful scientists work hard, and Mr. Smith is a scientist who works hard. Can we say from this that Mr. Smith is a successful scientist?
2. If all the angles in a polygon are equal, do the sides have to be equal?

Reductio ad absurdum method. An example attempted by all subjects was:

> In figure 1 we are told that AB is not parallel to CD, and we wish to show that p and q have different values. Complete the argu-

Fig. 1

Deductions. One question given to all subjects was taken from Lewis Carroll's *Symbolic Logic*. It read:

> Give the conclusion that follows from these three statements:
> 1. No potatoes of mine, that are new, have been boiled.
> 2. All my potatoes in this dish are fit to eat.
> 3. No unboiled potatoes of mine are fit to eat.

A more difficult question (consisting of two parts) given only to the fifth- and sixth-form pupils is given below. It is taken from the section on converses.

> What do we mean by the converse of a theorem? Give an example if you wish.
>
> What is the converse of: "If a quadrilateral is a rectangle, then its diagonals are equal"?

It will be appreciated that there were 50 questions in the two tests, with 1,538 answers to each question in test 1 and 1,418 answers to each question in test 2, quite apart from the answers obtained by the individual interrogation of 80 pupils. In order to reduce the amount of work, Reynolds analysed, in detail, the responses to 22 questions drawn from the six sections and from the two tests.

Results

To summarize the findings in limited space is a very difficult task. It will, perhaps, be best done by considering a question taken from each section. I must point out, however, that I have greatly simplified the findings—perhaps oversimplified them—and the percentages expressed below have been well rounded.

Generalisations

Let us consider the responses made to the example that involved Goldbach's conjecture. Some 40 percent of the first-, third-, and fifth-

late Piagetian theory into stimulus-response terms such as "internal reinforcements" and "transformation responses" is seen by Piaget as little more than a language game.

The second generation of studies reflects a dissatisfaction with the Piagetian position in two senses. First is the conviction that Piagetian stage theory places too rigid a limitation on logical thought acquisition. It is asserted by a number of investigators that the logical structures delineated by Piaget can be acquired earlier through various kinds of training than is implied by the Genevan "norms." In turn, Piaget is often critical of the (principally American and Soviet) "obsession" with accelerating logical thought development.

Second, many investigators are unwilling to accept the equilibration model as the explanation for logical thought acquisition nor the specific logical operations and structures said to be involved in particular kinds of thinking. In regard to conservation, for example, they do not accept the role of reversibility nor inversion and compensation strategies as necessary to successful performance. Nor are they willing to accept transitivity as necessary to seriation, or the classification logic as necessary to whole-part relations and class inclusion, and so on. Many critics undertake training studies for the purpose of exposing the "true" mechanisms of thought, or at least to reduce them to the simplest objectively observed constituents. In general, these studies can be divided into those concerned with the technology of training or learning and those concerned with the mecha-

subjects were always unaware of an infinite class (although this is very unlikely in the case of the younger children), but that they extended the generalisation only to instances close to the given ones. Around 10 percent of the first-year children answered in this way, decreasing to nil in the M sixth. All pupils who answered on the basis of "evidence sufficient" were clearly at a concrete-operational level of thinking in respect of this problem.

A third group of responses (averaging from 10 to 12 percent of the replies up to the fifth form, then decreasing to 7 percent) answered "Yes" but made use of the converse. Such children selected two odd prime numbers and by addition found an even one. Such a process, of course, always gives an even number, but there is no certainty that all even numbers will be determined by it. The answers revealed a failure to realise the direction of the given process and to give consideration to the universe of discourse. A few also glossed over the distinction between prime and odd numbers.

The percentage of pupils who accepted the generalisation, "with doubt," increased slowly up to the NM sixth form and then jumped suddenly to around 12 percent. Subjects realised that the evidence for the generalisation was insufficient; they looked on the generalisation as a hypothesis, although they made no attempt to test it, as might have been expected from Piaget's theoretical position.

Finally, there was a group of pupils who realised the lack of evidence for the generalisation. The percentage of answers of this type increased slowly with age until the sixth form, when it increased abruptly to somewhat over 20 percent. Subjects realised that the evidence from a number of positive instances could never be adequate to prove the generalisation for all even numbers. They emphasised the need for a more general proof before a conclusion could be reached. This more advanced view presupposes an adequate notion of a mathematical generalisation, that is, that it states a property of every element of a definite set. But it is interesting to note that not a single pupil made a complete test of the possibilities. If p is "x is an even number" and q is "x is expressible as the sum of two primes," the generalisation would be $p \rightarrow q$. No pupil considered testing the truth values of the four expressions

$$p \wedge q,\ p \wedge \sim q,\ \sim p \wedge q,\ \sim p \wedge \sim q.$$

The training studies will now be examined in some detail. It is a curious fact that from the entire range of interesting and important Piagetian experiments and observations, the most studied has been the conservation phenomenon. There is no doubt that conservation lends itself to easy experimentation, and that it is intimately tied to the general theory, but its attraction probably lies in the fact that it appears to be the case where a crucial test can be made of Piagetian theory. Many training studies are designed to make such a test.

Training Studies of Conservation

The ability to conserve is inherent in the development of quantitative concepts, since such concepts require the ability to maintain the invariance of the concept in spite of unrelated or related attribute transformations. In a recent statement of his ideas about conservation, Piaget ([c] 1968) emphasizes that conservation is possible only when there is a *composition* of quantitative variations which takes the form of a compensation of relations (higher X thinner = same amount), or an additive composition (nothing added, nothing taken away = same amount). The additive and compensatory compositions involve two types of reversibility, reversibility by inversion (a return to the original state), and reversibility by compensation, plus the identity operation (nothing added, or taken away). The identity operation occurs only in relation to the other operations and

While a partial test of these expressions may be expected of the twelve-to-fourteen age range, the full test might be expected in the fourteen-to-eighteen age range. Pupils who rejected the generalisation on lack of evidence showed a move to formal thought and a need for a general proof, but they did not test the hypothesis.

Symbols

We discuss the responses to the question "Is $-p$ negative?" Since $-p$ is not part of an algebraic expression, many children think of it as indicating a state (e.g., position on a number line) or as an operation (i.e., to subtract p from a certain quantity).

Of those who answered "Yes" to the question, the following percentages thought of p as indicating a state: around 28 percent in the first year, 55 to 60 percent in the third and fifth years, 45 percent in the NM sixth, and 18 percent in the M sixth. A typical reply was "$-p$ is less than 0; any minus quantity is negative." Those pupils who answer, effectively, "Yes, a state," have a restrictive universe of discourse for values of p, although they are not aware of this. Very few subjects (nil in the M sixth) thought of $-p$ as indicating an operation. Those who did concentrated on what the symbol indicated should be done to p and not on the range of values of p. Another group of pupils who answered "Yes"—mainly in the first and third years—appeared to think that p was a different kind of number. Their replies were of the type, "It does not exist," "It does not represent a number of articles." With these pupils the individual interviews clearly showed that a negative number was not regarded as a "real" number. Overall, the first-year pupils showed much uncertainty both in answering the question and in respect of their concept of negative numbers generally. Nearly one-third declined to attempt the exercise. Of those who did attempt it, many showed they had some familiarity with numerical terms like $+3$, -2 (e.g., as temperature) but their understanding was intuitive. This evidence is in line with the view that until the onset of formal-operational thought a child's grasp of negative numbers is intuitive.

There were, of course, pupils who responded in effect "not necessarily"; hardly anyone did so in the first year, but the ratio rose to around one-third in the fifth and NM sixth forms, and to three-quarters in the M sixth form. These pupils may not have explicitly decided to take the set of integers or the set of real numbers as their universe. Indeed, they may have considered only a very limited range of positive and negative integers for p, but their acceptance of some positive and negative integers for p showed a better appreciation of the position than those who considered only positive values. There were also some 10 percent of the subjects in

the first year who answered "No" but whose responses otherwise showed poorly developed concepts of negative numbers. In the other year groups there were very few replies of this type.

Assumptions

The responses to the question "What do we mean by a hypothesis?" were divided into those which showed some appreciation of a hypothesis and those in which the ideas expressed were of little value. Examples of good answers were of the type "Something supposed" and "Something taken for the sake of argument." Pupils whose replies fell in this category gave the impression that they regarded a hypothesis as a statement tentatively accepted and to be used as a basis for reasoning. Very few responses in the first and third years were of this kind, the percentage rising to around 15 in the fifth year, and to about 40 in the NM, and to a little over 50 in the M, sixth forms.

Nearly all the answers in the poor category centered around one or more of four possibilities: a hypothesis is a true statement, an untrue statement, a proved statement, or one that cannot be proved. The percentage of poor answers varied from 5 percent, in years one and three, to some 25 in the NM, and to 20 in the M, sixth forms. It will be appreciated from the figures given above that in each of years one and three over 90 percent of the pupils did not respond to the question, this figure dropping to 20 percent in the M sixth form.

Converse

Here we discuss the replies to the exercise: "All successful scientists work hard, and Mr. Smith is a scientist who works hard. Can we say from this that Mr. Smith is a successful scientist?"

Some 25 percent of the responses of the first- and third-form pupils answered "Yes" in one form or another. This figure decreased in the fifth form, until in the M sixth form only 1 to 2 percent answered in this way. None of the subjects who so responded realised that they had accepted the converse of "all successful scientists work hard" or had equated the set of successful scientists with the set of hard-working scientists. A typical reply was "Mr. Smith works hard and those scientists who work hard are successful." This kind of reply regarded as empty the set of scientists who work hard and are not successful. Other pupils who put "Yes" regarded the situation as one reflecting real-life conditions rather than as a hypothetical one. Their view was that if you worked hard enough, then in the end you should be successful; and the answer "Yes" was given in spite of exceptions.

There were two categories of reasons for "No." In the one a good

reason was given; in the other the reason advanced was not adequate. Those pupils in the "No, good reason" category realised that it could not be said with certainty that all hard-working scientists were successful. Yet even though the pupils understood that the converse of the first statement of the exercise was not necessarily true, a few supported their conclusion by reference to real-life situations and argued that other factors such as intelligence or skill might well affect success. These children, too, were looking upon the situation as a practical one. However, around 55 percent of pupils in the first and third forms gave a "No, good reason" type of reply, the figure rising to 85 to 90 percent in the NM, and to over 90 percent in the M sixth form. About 18 percent of first- and third-form pupils gave a "No, poor reason" type of response, but among the older pupils the numbers of such replies were negligible.

This question involved the inclusion relation between two classes: the class of successful scientists X, and the class of those who work hard Y. The relation was $X \subseteq Y$; if X' is the complement of X in Y, then $X + X' = Y$. So the problem was concerned with three classes, X, X', Y, connected by $X + X' = Y$. This type of problem can, of course, be solved at the stage of concrete-operational thought. But two features make it difficult to solve it at this stage of thought. First, the situation is not easily imageable, and since it is certainly not perceptible it is not easily intuitable. Second, X' was not explicitly mentioned. One would expect, therefore, a good percentage in first and third forms giving a "No, good reason" type of reply, and a rapid increase with the development of advanced formal thinking. This is broadly what was found.

Reductio ad absurdum method

Here we discuss the responses to the exercise: "In figure 1 we are told that AB is not parallel to CD, and we wish to show that p and q have different values. Complete the argument that starts, 'Either p and q are equal or they are different; if p and q are equal then. . . .' "

In the schools used in this investigation, parallel lines were studied in the first forms, and all pupils knew that when parallel lines are cut by a transversal the alternate angles are equal and the corresponding angles are equal.

The responses fell into four categories. The first category was poor; in this no effective contribution was made to the argument. There were comments on p or q or on the diagram (e.g., "p and q are alternate" or "CB is a transversal of AB and CD"), but no attempt was made at any argument. The percentage of pupils responding in this general way declined from some 25 in the first form to 6 in the fifth form, and nil thereafter.

Category two included the replies in which it was deduced that AB was parallel to CD from the hypothesis $p = q$. Some pupils whose responses were put in this category made the other unjustified assumption; that is, "As p and q are not equal, lines AB and CD are not parallel." The number of pupils in this category steadily declined from around 35 percent in the first year to 14 percent in the M sixth form. In general the pupils whose replies were so categorised failed to realise the role of the original hypothesis, $p = q$, which became the source of a deduction that contradicted the data. They were unable to compare their conclusion with the data that AB and CD are not parallel. It is also worth noting that over half the first-form pupils interviewed said that alternate angles were always equal.

The third category of responses was termed *contradiction reached*, for the pupils indicated that they had arrived at a contradiction but did not know how to deal with it. Put in another way, all the pupils realised that a contradiction had been reached but they did not see that the source of it was the hypothesis $p = q$. A typical reply was "AB parallel to CD, which is not so." Remarkably, the percentage of pupils giving this kind of response remained steady and ranged only from 12 percent to 18 percent across the forms.

The percentage of answers in the fourth, or good, category increased from 14 in form one, to 38 in form three, to around 60 in the fifth and NM sixth forms, to close to 70 in the M sixth form. All the pupils whose responses fell into this classification recognised that the deduction $AB \parallel CD$ was inconsistent with the data and realised that the source of the contradiction was the hypothesis $p = q$. So this was discarded and the only other alternate $p \neq q$ accepted.

Deductions

Responses to the potato question taken from Carroll's logic are now examined. Five types of answer were found:-those correct, those that did not reach a solution but that showed no inconsistency, those that made a contradiction, those that added new information, and those that treated the three statements as unrelated.

The percentage of correct responses increased from around 32 in form one to 75 to 80 in the M sixth form. There was a remarkably steady percentage of replies in the second category—around 25 up to and including the NM sixth form, thereafter the figure fell to 9. Examples of these incomplete replies were "New potatoes are unfit to eat"; "The potatoes in the dish are boiled"; "No new unboiled potatoes are fit to eat." For ease of discussion let us consider these categories of response first. To reach a solution a number of steps are necessary. From the three

statements a suitable pair must be selected and a deduction made. This deduction must now be regarded as a statement about the potatoes, and combined with the unused statement of the original three to draw the final conclusion. Apart from a few answers in the second category which gave one of the original statements in another form (e.g., "All my unboiled potatoes are not fit to eat") no one had any difficulty in reaching the deduction from one of the suitable pairs of statements. But two difficulties appeared to hinder the solution. First, the deduction was regarded as having to be combined with the unused statement of the original three. Second, the combination of the deduction and the second statement (of the three) seemed easier when the deduction was in the form "My potatoes which are fit to eat are not new" than in the form "All my new potatoes are unfit to eat."

When we turn to the other categories of response, we find that some 15 percent of pupils in years one and three provided a contradiction, the corresponding figures for the other years becoming very small. In a very few instances in each age group new information was added (category four type of response), while around 13 percent in the first year but very few thereafter treated the statements as unrelated (category-five response).

The responses that yielded the contradictions are interesting; all gave a reply inconsistent with the data or with a deduction from them. Statements one and three are both in negative form; i.e., one set was said to have no element in common with another set. Contradictions such as "The new potatoes are boiled" and "All my potatoes are boiled and fit to eat" were attempts to put these statements in a positive form. Other responses in this category involved errors regarding the age and state of the potatoes (e.g., "The new potatoes are fit to eat"). A common failing was the lack of check between the answer and the data given.

THE RESULTS IN THE LIGHT OF PIAGET'S DEVELOPMENTAL SYSTEM

Since all the subjects were above average in attainment and measured intelligence, it is reasonable to suppose that those in the first and third forms would be acquiring formal-operational thought and be at Piaget's stage IIIA. Those in the fifth and sixth forms would be aged 15+ to 18+ years and might be expected to make full use of formal-operational thought and be at Piaget's stage IIIB. However, because of individual differences in ability which would make some very bright third formers more intellectually advanced than some fifth formers, some overlap in respect of performance must be expected. Indeed, this is what happened.

for there were common approaches to the questions in all age groups. But the occasions of notable improvement in the fifths' and sixths' answers in the questions we listed were:

1. In the questions dealing with generalisations
   Here there was a clear move to emphasise the lack of evidence for a generalisation based on a number of instances. and the increased percentage of the replies that indicated the generalisation $2^n > 2n + 1, n \geqslant 3$.
2. The emphasis · the value of symbols for generalising
3. The better understanding of the meaning of *hypothesis* and *logical statement*
4. In the percentages answering correctly the exercise relating to Mr. Smith and the hard-working scientists in the section on converses
5. The increase in the percentage of responses in the good category to the problem based on the reductio ad absurdum method
6. In the percentages giving a correct response to the problem taken from Lewis Carroll's *Symbolic Logic* in the deduction section

However, to some questions the answers of the fifth- and sixth-form pupils showed only a gradual improvement over those in the first and third forms. Nevertheless Piaget's formulations regarding stages of thinking account for a good deal in the nature of the replies. Answers that were characteristic of the concrete-operational stage of thinking appeared regularly. but the answers also indicated an increasing ability to use formal-operational thought with age.

Some Points for Consideration

The questions posed to the subjects showed considerable variation in the degree of their structure. from the fully structured. precise situation involved in the potato question to the following situation (presented to all pupils):

> If all the men who are on Committee A are put on Committee B, which of the following statements are true. false, or cannot be decided?
>
> 1 Every man who is on B is on A.
> 2. Any man who is · $\,$or B cannot be on A.
> 3. Every woma · on B is not on A.

In this problem only · t is given.

Reynolds argues that there are a number of reasons why the neat

theoretical framework in respect of formal-operational thought arising out of Inhelder and Piaget's experimental tasks did not explain all the findings in this study. He maintains, for example, that the degree of structure of the problem is relevant. The advantage of a well-structured problem lies in the fact that its assumptions, variables, and universes of discourse of the variables are easily identifiable. The problem solver then has no need to introduce assumptions or hypotheses from outside the situation as he attempts to solve the problem. He can construct hypotheses by relating in various ways the variables of the situation, deduce the consequences of the hypotheses, and then test them.

In a loosely structured problem either all the assumptions are not stated or the variables are not easily identifiable or the universes of discourse of all the variables are not given. To such a problem the solver must bring his own assumptions and universes of discourse drawn from his own experience. For example, in the question that asked whether $-p$ was always negative, those who replied, in effect, "Yes, a state," assumed that p took positive values; but the fact that this caused a restriction was not recognised.

In the question relating to the committee in the deduction section, some of the assumptions made were:

No woman is on both A and B.
There are no women on A.
There are no women on A or B.
After the change there is only one committee.
Some women on B can be on A.

These assumptions gave a more definite structure to the situation and led to erroneous conclusions. Reynolds argues that in Inhelder's experiments a pupil had a better opportunity to rectify a wrong assumption by practical manipulation. We must also note, however, that Reynolds does not make clear any differences he found between the written answers and the answers obtained by oral questioning: in the latter instance supplementary questions could have helped the subject to see the consequences of his assumptions and so arrive at inconsistencies. I personally feel that, throughout the study, pupils would perform rather better at the individually administered tasks than on the written tests, and I think Reynolds would agree.

So much for the structure of the questions. When we turn to the subject's knowledge of the concepts involved, we find, as in all other studies, that the level of his understanding of mathematical concepts has an effect on the quality of his answers. There is much in common here with Gagné's viewpoint, namely, that strategies of thought for use by the

HARRY BEILIN

The Training and Acquisition of Logical Operations

Mathematicians and mathematics educators are interested in Piaget's theory of cognitive development because it explains how mental operations basic to mathematical thought develop. They do not, on the whole, have much interest in Piaget's views of fundamental logical or mathematical relations, such as his ideas about the logical properties of number. Since, according to a number of philosophers of science, it is desirable to isolate philosophic and logical systems from psychologizing, mathematicians and logicians are able to view the psychological implications of Piaget's theory quite independently from its mathematics, even though a significant part of the psychological theory has mathematical and logical content.

Beyond this, interest in Piaget's theory is centered on two of its features. First is the identification of the functional and structural properties of thought as they undergo change with age. The theory holds that while adult forms of thought have their precursors in the structures of child thought, they are qualitatively different from the thought of earlier periods. The early forms are not suited to particular kinds of problem solving. Problems that entail the use of propositional logic, for example, are approached by children with strategies that lead to immature and incorrect solutions. This view stands in opposition to theories of thinking and development that assume that cognitive processes are the same for

Preparation of this paper was aided in part by NICHD Grant HD-00925-09-10.

The author is grateful to Joan Kay Clayton and Carol Doyle for their assistance in the preparation of the bibliographic materials.

all ages, with the child's thought only a less complex and quantitatively reduced form of the adult's.

The second aspect of the theory that is of theoretical and practical import is the position it espouses vis-a-vis learning. Piaget's theory is a developmental theory that subsumes learning to development, in contrast with behavioristic theories that explain development in terms of the processes of learning. Piaget's view is that experience results in learning only to the extent that the elements of experience are assimilable to existing cognitive structure. Experience, whether it involves practice, reinforcement, need reduction, or verbal rule learning, yields no persisting residue if it does not take place in the context of appropriately available intellectual resources. This is a very strong claim and it has not gone unchallenged. It is the intent of this review to examine a variety of studies that have questioned, defended, or examined the Piagetian thesis concerning the relation of learning to logical thought development.

In Piaget's conception of cognitive development there is continuous change in which the child's thought emerges out of the actions he performs upon the objects in the world about him. These actions upon objects constitute the model for later thinking, since the most important element in both the child's and adult's thought processes, the logical "operations," are associated with action. Logical thought is conceived as a form of implicit action. Action is represented in the general property called "reversibility," a form of action that can be canceled by a reverse action. In its logical form, an operation can be canceled by an inverse or compensating operation, as in the case where the addition of a unit is canceled by the subtraction of the same unit. This characteristic of thought is embodied in all logical thought, and for the Genevans is critical in determining true or complete "operativity," that is, the full achievement of a logical thought system. The flexibility of operations represented by reversibility is the feature that distinguishes operative from nonoperative thought. It is a flexibility, nonetheless, that develops within a structured system organized in accord with logical principles that are codified by Piaget as the logic of classification, the logic of relations, and the propositional logic. Reversibility and concrete-operational thinking develop at about the age of six or seven years. Preoperative thought, which exists from the end of the sensorimotor period (about the age of eighteen months to two years), while not without its logical properties, is at most a limited logic (a semilogic, as Piaget puts it), and it lacks the key reversibility feature. The entire system of logical thought develops under the control of a self-regulating mechanism which Piaget denotes as equilibration. Equilibration, operating in conjunction with maturation and experience, is the central mechanism by which development occurs. It is the process by

virtue of which the individual constructs logical schemes out of the elements of his experience as well as from already constructed operations.

The Genevans first became interested in learning (or training) studies as a defense of the equilibration model. The most important theoretical alternative as they saw it was the general behavioristic model common to Hull, Skinner, Pavlov, and others. Whatever other features they might stress, such as need reduction, contiguity, or feedback, they have in common the attribution of learning to the external reinforcement of responses made by the subject. For Piaget, learning in this sense is "provoked by situations." It is provoked as opposed to being spontaneous, which is the prime characteristic of development. The Genevans felt it necessary to attack the behavioristic alternative to equilibration theory because of its appeal to learning theory as the principal alternative explanation for the phenomena of development.

Among the training studies that followed one can delineate three generations of researches. The first was by the Piagetians themselves, the second by those with a variety of theoretical orientations and methods. The third generation reflects a return to the training studies by the Genevans. In the first generation an attempt was made, as indicated, to buttress the equilibration model against behaviorist attack. The studies by Smedslund ([a] 1959; [b] 1961), Wohlwill ([a] 1959), and the Genevans themselves (Piaget [a] 1959) are significant. Smedslund contrasted external reinforcement training with conflict-equilibration training and showed that while reinforcement might be effective for learning it did not compare with the effectiveness of a conflict-equilibration procedure. He also emphasized that when learning did occur it was with subjects who already had the rudiments of operational structures available to them. Piaget ([b] 1964), in commenting on the Smedslund studies, observed that while Smedslund was successful in inducing weight conservation with his method, he was not successful with transitivity training with the same method. This led Piaget to make a distinction between training physical relations and training logicomathematical relations. He argued that training could be successful for physical experience but not for the construction of logicomathematical structures. He also took note of what he saw as Wohlwill's success in inducing number conservation through additive operations. He cited this to be an example of learning when one bases a more complex structure on simpler structures if there is a natural (i.e., logical) relation between them. What is common to Smedslund's, Wohlwill's, and other Piagetians' experiments in this area is the rejection of "external reinforcement" as a model for the acquisition of logical and infralogical structures.

In this period Piaget also rejected attempts at a theoretical rapprochement with neoassociationism. The effort by Berlyne (e.g., 1965) to trans-

demonstrate the validity of this argument, they conducted a perceptual screening experiment with four-, five-, six-, and seven-year-old children. The essence of the experiment involved requiring the child to make decisions about the quantity of water poured from one container to another in a variety of conditions, with and without screens. The child made comparisons between the predicted height of the water and the actual height. The purpose of the screen was to force the child to base his judgments upon the identity of the water, which would help him "resist" the effects of changes in the appearance of the water. Resistance to the alternations in water shape was considered to result from the linguistic representation of the perceptual relationships viewed in the experiment. The effect of the training by screening was shown to be considerable in the five-, six-, and seven-year-old groups. The four-year-olds, however, showed no improvement in the transfer posttest.

Piaget ([c] 1968) is specifically critical of these studies. First, he implies that Bruner's subjects attained a type of pseudoconservation, whose true status would be exposed by the use of a simple control procedure. Piaget is also critical of Bruner's conception of logical compensation, suggesting that Bruner fails to distinguish "functional covariation" from "operational compensation," as well as failing to distinguish "reversibility" (which is logical and operative) from "empirical return" (which is a "physical notion"). His most important observations, however, are made in regard to identity. Piaget elaborates a developmental sequence for identity in which a preoperative type of identity progressively yields to an operative identity that reaches maturity concurrently with the related conservation operations. Preoperational identity which Bruner is addressing, says Piaget, can only lead to pseudoconservation. Piaget is additionally quite critical of Bruner's linguistic argument, citing evidence that language is subordinate to operations and "does not constitute the formative mechanism of the operations" (Piaget [c] 1968, p. 33).

The effects of Bruner's screening procedure have been interpreted in a broader context elsewhere (Beilin [c] 1969). The linguistic forms in which the perceptual data are coded are seen to encapsulate the rules governing the conservation operations. The statements thus act as algorithms for the processing of perceptual input. This leads, as a rule, to limited classes of correct solution that lack the flexibility of true reversible operations. (More will be said on this score later.) In any case, it is evident that a substantial number of nonconservers from age five on can be induced to conserve with the screening procedure, although the fact that four-year-olds were not able to profit from the procedure is not adequately accounted for by Bruner, even though four-year-olds have a sophisticated linguistic system available to them.

In an attempt to systematically formalize the distinction between identity and equivalence conservation, Northman and Gruen (1970) tested children on a variety of tasks involving identity and equivalence procedures. They were not able to substantiate their prediction that identity conservation would precede equivalence conservation, although they did find that transitivity emerges at about the same time as the operations necessary for conservation.

Although the so-called identity and equivalence tasks represent two different ways of testing conservation, the mechanisms underlying them are not substantially different. An understanding of transitivity would seem to be required in the equivalence conservation test, since one of the elements is used as a common measure. Since the common measure is transformed or relocated, however, it would seem logically required that conservation is a requisite for transitivity rather than transitivity for conservation. In addition, if the identity notion undergoes operational development in the way Piaget suggests, then the reduction of conservation to a single identity mechanism would seem inadequate, particularly if this form of identity is akin to a notion of object constancy.

The attempt to reduce the notion of conservation to identity relates to another problem which pervades the conservation literature. It involves the definition of conservation. With an appropriate definition, conservation can be demonstrated at a very early age, as Bruner and others (1966) and Mehler and Bever (1967) attempt to do. If conservation is defined closer to Piaget's meaning, however, it is seen as a later emerging achievement. Gruen ([b] 1966) shows this to be true in the use of weak and strong criteria for evaluating a subject's conservation responses. A conceptual analysis of the conservation notion (Beilin [c] 1969) reveals that certain uses made of the term, such as Bruner's, distort the meaning and significance of the phenomenon. The association of "same" in relation to "number," for example, involves the conceptual attributes of number. Dealing with the conceptual attributes of number requires cognitive capacities in the child which are different in kind from those in which "same" is used in relation to the notion of "object" (e.g., water). The cognitive mechanics needed to deal with "same number" are more sophisticated than those required for dealing with "same water." The hierarchical relation between the concepts is paralleled by a hierarchical relation between the thought processes required to conceptualize them. Attempts at reduction from one level to another on a logical basis alone are not likely to succeed. Attempts (such as those of Brainerd and Allen 1971) to show that the criterion problem is not a significant one are misleading, particularly if the argument is that both successful and unsuccessful training studies have used stringent criteria for assessing conservation perform-

ance. Stringent operational criteria are not the same as demanding conceptual criteria, and the failure to distinguish them is a serious oversight.

While some studies involve what appear to be conceptual or theoretical differences from the Piagetian model, others treat experimental issues. The Mehler and Bever studies involve both and represent instances in which Piaget's theory concerning conservation is vigorously refuted with ostensive conservation experiments (Mehler and Bever 1967; Bever, Mehler, and Epstein 1968). In response to critical reaction, however, these are later identified as something other than conservation (Mehler and Bever 1967; Beilin [b] 1968; Bever, Mehler, and Epstein 1968; Piaget [d] 1968).

Cognitive conflict

Another series of training studies deals with an acquisition model that also derives from Piagetian theory. These training studies start with the work of Smedslund ([a] 1959; [b] 1961), who posits that cognitive change occurs from the conflict between strategies or schemes that are constructed out of the child's experience. The conflict is not between an existing scheme and data from perceptual or sensory experience but between ideas themselves, that is, between an existing scheme and one newly developed from experience.

Smedslund uses two methods to create conflict: a deformation procedure, whereby a transformation occurs through a deformation of the object or its location, and an addition/subtraction (A/S) procedure, whereby additions to and subtractions from an object or an array of objects are made. Smedslund's studies show that when the procedures are effective in changing nonconservers into conservers (and they are not always successful) it is usually when there is already some evidence of conservation in the child's performance. The test he makes of the effectiveness of other procedures to induce conservation (particularly reinforced practice) shows them to be ineffective. Other investigators are divided on the efficacy of Smedslund's conflict methods. Beilin ([a] 1965), who tested the deformation procedure, found it to be ineffective; and Smith (1968), who used the A/S procedure, found it to be equally unproductive. Mermelstein and Meyer (1969) obtained no significant changes in performance with a procedure that purports to be an approximation to a deformation procedure, although they obtained no positive results with any other training procedure.

A study by Winer (1968) conceives of the A/S procedure, following the proposal by Wohlwill and Lowe (1962), as a set-training procedure that leads to the development of inferences. He finds that A/S training is effective in improving conservation, but "conflict trials" in which addition/

subtraction is tied to a deformation procedure were even more effective. In spite of this finding, Winer considers the effect of the conflict procedure to be minimal. One may with equal justification, however, consider the results as supporting the conflict interpretation. Smedslund ([d] 1963), in training for conservation of length, obtained the greatest gains from the use of a method that required subject anticipations of object displacements (a deformation procedure), whereas he found that increasing illusion effects with the Muller-Lyer illusion led to the smallest gains. Murray (1968), who also used the Muller-Lyer illusion to create cognitive conflict, did obtain significant increases in conservation compared with a control. Conservation acquisition did not transfer to an area conservation task, however, and as would be expected, older *Ss* showed greater gains in conservation than younger *Ss*. Wohlwill and Lowe (1962), employing an A/S procedure, obtained little increase in conservation performance. In interpreting the experiment's dynamics they hold, however, that the A/S procedure leads to the child's development of an inference from the contrast of the A/S condition with the condition without A/S. In the Genevan sense two schemes are being contrasted by the procedure, and the resulting conflict leads to a new organization of schemes. The inference hypothesis, on the other hand, implies a process of induction, the nature of which is as little understood as that of schema construction, although it is more often identified with behavioristic interpretations.

Gruen ([a] 1965), using the Wohlwill and Lowe A/S apparatus and procedure, found that *Ss* given conflict training outperformed *Ss* given direct training on the apparatus in which there was no A/S procedure. He found very little transfer, however, from number training to length and substance conservation.

In spite of the few negative studies, the conflict procedure appears capable of leading to improved conservation performance. Addition/subtraction is superior to the deformation procedure, even though Smedslund found the effects of deformation demonstrations to be superior and Wohlwill himself felt A/S training to be of relatively little help except for implicitly demonstrating reversibility. Conflict training is more effective with older children and does not transfer to types of conservation not trained. Another finding which appears consistently in these studies is that the conservation of discontinuous quantities (number) is achieved prior to other types of conservation.

Reversibility

Piaget, as indicated, puts great stress on reversibility as the key to conservation. In a recent discussion of the subject (Piaget [c] 1968), reversibility is associated with both inversion and compensation strategies

in conservation thinking. The earlier version of the theory, which still placed considerable stress on reversibility, prompted Wallach and Sprott (1964) to directly train for reversibility as a means of promoting conservation. First-grade children were given training designed to show, through the reversibility of rearrangements of order, the invariance of number. This was done by showing that a fixed number of dolls could be fitted back into their beds after they had been removed and either spread out or bunched together. In a later experiment (Wallach, Wall, and Anderson 1967), however, children did not necessarily conserve even though fully cognizant of reversibility (Wallach 1969). This led Wallach to question whether training really induced reversibility or merely led the child to recognize that a misleading cue was in fact misleading, a fact that is not sufficient for conservation.

A study by Roll (1970) shows that reversibility training does lead to improved conservation performance compared with that of a control, as does a study by Brison (1966). The latter does not identify his procedure as a reversibility procedure, although it utilizes transformations of liquids from differently proportioned jars and then retransformations (reversal) to the original jars. The liquids in the original jars are unequal, so the conservation is ostensibly one of inequalities. Brison reports transfer to conservation of substance. A study by Carey and Steffe (1968) that has reversibility training as the key to its instructional procedures reports increases in both conservation of equalities and inequalities as a function of the training.

It appears from this limited number of studies that the reversibility procedure itself is capable of inducing conservation. The subject verbalization data of the Wallach study, however, make it unclear as to what is occurring in the training task. The reversibility training "method" may not be leading to the construction of a reversibility mechanism even though it leads to improved conservation *performance.*

Learning studies

The prior sections have in common the fact that the mechanisms proposed to be basic to the acquisition of conservation are at least in some sense within Piagetian theory. The training studies to be cited here go outside the Piagetian explanatory system for at least part of their logic of justification. In these studies an attempt is made to foster the acquisition of conservation by (1) training the child to disregard or ignore misleading perceptual cues, (2) training the child to attend to the relevant attribute (with or without learning to ignore "irrelevant" cues), and (3) training the subject to differentiate the "real" from the "apparent."

The method of choice in these studies usually involves the use of a con-

cept-formation procedure based on a discrimination model in which discrimination and concept acquisition is achieved by the reinforcement of correct responses. These procedures are usually associated with behavioristically oriented research, although the researcher who uses these methods may be neutral or even negative in his regard for behavioristic theory. It may be held, though, that the very use of a reinforcement procedure commits the researcher implicitly to a behavioristic interpretation of learning and development.

The discrimination-based concept-formation procedure is identified quite differently in different studies. With slight modification, for example, it becomes training for "learning sets."

An explanation of conservation acquisition in terms of "set" receives early expression in Zimiles (1963). Conservation is interpreted as a tendency to respond with consistency to a conceptual property (e.g., number) rather than according to other (e.g., spatial) criteria. Zimiles, however, undertakes no research to substantiate this thesis. Attempts to train children by drawing attention to the relevant dimension in a conservation task with a reinforcement procedure have been reported both as unsuccessful (Beilin 1965; Smedslund [b: II, III, V, VI] 1961; Hatano and Suga 1969; Smith 1968) and as successful (Kingsley and Hall 1967; Gelman 1969; Eull and Silverman 1970). Kingsley and Hall (1967), in a study conceptually grounded in Gagné's (behavioristic) theory of concept acquisition, used a learning set procedure for training weight and length conservation with Smedslund's extinction procedure as a test of the achievement of conservation. There was a significant improvement in performance, including transfer to a conservation of substance task. However, only 3 of 17 trained *Ss* who achieved conservation resisted extinction, and no natural conserver resisted extinction. Hall and Kingsley (1968) emphasize in another study that experimental conditions have an important bearing on the outcome of conservation experiments. They particularly identify the role of visual cues, verbal instructions, and labels in affecting conservation performance. They also show that it is possible to obtain extinction of conservation among college students, which contradicts some of Smedslund's assumptions about the "counter-suggestion" or extinction procedure. These findings and similar results of others place in doubt the extinction procedure as a strong test of operativity. Gelman (1969) holds that the failure of children to conserve is due to inattention to relevant qualitative attributes or to attention to the irrelevant features of a display. The theoretical basis for conservation and other conceptual learning situations is interpreted to be associated with the function of attention. Gelman provided learning set training with feedback in both length and number conservation. The training consisted of 32 six-trial

oddity problems employing three stimulus objects. Half of the series varied in number and half in length. There was within- and between-problem variation in color, size, shape, starting arrangement, and combinations of quantity. The results indicate that learning was effectively achieved. There was almost perfect specific transfer (i.e., to the same type of task for which the *S* was trained), and about 55 to 58 percent nonspecific transfer to liquid and mass conservation. Finding a large amount of nonspecific transfer is most striking. Another unusual aspect of the study is the report of "correct" verbal justification of conservation following an ostensively nonverbal training procedure. A study that pursues the possibility that Gelman's learners were pseudoconservers (Eull and Silverman 1970) modified the posttest procedure so that it would differentiate subjects who could make a conservation judgment from those who could make what I have referred to as a quasi-conservation judgment (where there is no object transformation). The authors confirm the general effects of training but obtain much less learning than Gelman found. They also report many correct verbalizations in *Ss* who are unable to give a correct nonverbal conservation response in the experiment.

A pervasive difficulty in this area of research is that different investigators conceive the same training techniques to be relevant to different mechanisms and as supporting different interpretations of the conservation phenomenon. This is exemplified in the research of Halford and Fullerton (1970), who use the same "reversibility" procedure as Wallach but describe training in terms of "discrimination" and "sets." They report positive, but not unequivocal, results of training. In the posttest, an inequality "set" is responded to correctly by more conservation-failing control subjects than conservation-failing experimental subjects—a fact that is explained by the dubious observation that conservation of inequality is easier than conservation of equality when the order of difficulty is usually the reverse (Beilin [c] 1969). The authors attribute success of training not to reversibility but to "sets" induced by the discrimination procedure.

Reinforcement effects have been tested in a number of investigations. A recent study of weight conservation (Overbeck and Schwartz 1970) compares the effects of reinforcement with passive and active subject participation. In the process it exposes a phenomenon common to many conservation training studies. It is rarely clear from the report of these experiments as to how much verbal interaction has taken place between experimenter and subject, except in those studies that make a point of measuring the effect of verbal variables. In the Overbeck and Schwartz study, reinforcement is identified as verbal feedback for correct and incorrect responses, and also by the provision of a verbal rule appropriate

to the problem. Verbal rule instruction, however, is more than a reinforcement procedure. It becomes extremely difficult, then, to clearly identify the experimenter's manipulations and determine what may be causally related to experiment outcomes. Reinforcement is reported as facilitating performance, with no differential effects for active or passive participation. The authors make the general observation that the procedures that apparently succeed in training conservation of weight have one type of "reinforcement" in common: verbal rule instruction!

Using a method in which a blindfolded subject drops equal numbers of beads into two jars, Feigenbaum and Sulkin (1964) show significant improvement in performance compared with a "reinforcement by addition and subtraction method." This approach to the "reduction of irrelevant stimuli" is interpreted by the authors as supportive of equilibration theory and not of stimulus-response formulations, since the active experience of manipulating the beads is considered to be decisive. A study (Hatano and Suga 1969) that emphasizes practice and reinforcement with conflict and negation procedures concludes that relatively little learning occurs and then only in subjects who early in training show evidence of conservation. It proposes that number conservation requires discriminative response to transformation with the classification and coding of appropriate responses and the elimination of perceptual cues. Peters (1970) interprets the mechanisms underlying the transition from nonconservation to conservation to be associated with cue discrimination and verbal mediation. He finds that "perceptually guided cue discrimination" training leads to significant improvement in conservation performance.

Another series of studies bases training on the predication that measurement strategies are functionally related to the development of conservation abilities. Wohlwill ([c] 1970) has made this point together with the assertion that the success of measurement procedures is due to their "alerting" the subject to the dimensional features of conservation. This leads the child to respond in a "more conceptual" or "more differentiated" fashion. Bearison (1969) reports a study within this framework in which measurement procedures based on a counting strategy (with beakers) led to improved performance in a conservation of continuous quantity task which transferred to area, mass, quantity, number, and length tasks. A group of Soviet studies are also based on the use of measurement strategies. The theoretical framework of these studies employs concepts very similar to those employed by Gagné in which task analyses of a hierarchical nature are developed. They also employ notions similar to those involving learning sets, but the theory also goes far beyond these notions. The Soviet position, which like Piaget's proceeds from the assumption that action is the central problem of psychology, differs in important ways from

Piaget's (Galperin 1966). One of these is in the emphasis put on the orienting mechanism and its construction. The orienting mechanism has two components: the basic conceptual scheme of the real-world phenomenon, and the *algorithm* of actions involved in identifying or reproducing this scheme. Experimenter-constructed models or instructional strategies are given to the child himself before he begins to learn to serve as a guide to his actions. They also act as a guide to the trainer. In the case of conservation, the models are based on the use of measurement markers and the training is divided into three parts: (1) the use of markers for the comparison of two pluralities, (2) the comparison of size of two continuous magnitudes with the help of a third object, and (3) measurement itself with the application of different measures. The goal of this procedure is to help the child create an "indirect plan of thinking," which is then transferred to the concept being trained (Obukhova 1966). The training procedure is said to be very successful, although detailed data are not reported.

A number of training studies based on one form or another of behavioristic theory and emphasizing an associated experimental methodology have been successful in demonstrating acquisition of conservation, particularly when conducted by researchers sympathetic to these theories. Those studies that stress reinforcement (with the implication that some sort of associative bonding leads to conservation) are not so successful, although this approach seems to have been used mostly by those who intend to reject it. The studies that are more successful are those that emphasize the acquisition of "learning sets," where the learner acquires the ability to attend to "relevant" dimensions and ignore "irrelevant" dimensions. This approach, however, is relatively neutral with regard to the mechanisms of conservation. An equilibration theorist would also maintain that the conserving child attends to the relevant dimensions (or attributes) and ignores the irrelevant ones (i.e., distracting perceptual cues). It is in fact one way of defining "decentration." "Attention" is not a sufficient explanation of what occurs in conservation, even though it is a necessary ingredient to an adequate explanation. The theoretical differences in approach to the attentional notion are those associated with what one considers attention to be in the service of. Behaviorists like Zeaman and House (1963) hold that attention permits associative learning processes to come into play in concept acquisition. Other behaviorists may be inclined to view attention as leading to the use of "inference processes." An equilibration theorist, on the other hand, holds that attention is in the service of cognitive operations. Attention in this sense is part of a performance model of acquisition and not a competence model, to use an analogy from generative-transformational linguistics. Although

these learning studies attempt in the main to illuminate the mechanisms of conservation, they do not succeed in doing so because even where success is demonstrated, as in the learning set experiments, they do not constitute crucial experiments that confirm one theoretical explanation against all others. In addition, alternative explanations are usually possible and equally plausible for the same results. In the same manner that behaviorists use Piagetian procedures but explain them in their own terms, so is it possible to take a behavioristic experimental procedure and explain the results in cognitive and equilibration terms. In addition, the fact that achievement results from a particular training procedure (such as the "learning set"-reinforcement paradigm) does not ensure that the mechanisms of acquisition are isomorphic with those of the procedure or that the procedure identifies the necessary and sufficient conditions for acquisition. Clearly no procedure discussed so far can claim such a distinction.

Verbal training methods

A large number of investigators claim that Piaget slights language in conservation acquisition. Consequently, a series of investigations have sought to determine the role of verbal elements in the process of acquisition. One approach assesses the role of lexical terms, such as large-small, wide-narrow, and so on, in conservation concept acquisition. A second approach assesses the effectiveness of verbal as against nonverbal reinforcements. A third determines the efficacy of verbal rule instructional procedures. An approach of this type was used by the present writer (Beilin [a] 1965); the rules that govern conservation performance were represented in verbal form and coordinated with the manipulation of appropriate materials. The rules could represent reversibility by inversion, reversibility by compensation, and the identity operation. The power of the procedure was assessed in comparison with a deformation equilibration procedure, an attentional-orientation method, a concept attainment procedure based on a discrimination model (i.e., a nonverbal "learning set" procedure), and against the performance of a control. Verbal rule instruction was the most effective in inducing conservation over that of the other methods. There was little transfer, however, to a task for which the subjects were not trained (a quasi-conservation task), although there was appreciable "transfer" among natural conservers.

The verbal rule instructional (VRI) procedure was tested by Smith (1968) in a conservation of weight study (Beilin trained conservation of length and number), and the efficacy of the procedure was again shown. This time it was pitted against a reinforced practice method and a Smedslund type of A/S procedure. Only the VRI groups showed significantly better performance than the control.

The Overbeck and Schwartz study (1970) already cited also used verbal rule instruction, but the technique was so confounded with other procedures that it is difficult to assess its effect, although it did lead to improved conservation performance. The study by Peters (1970) also confirms the effectiveness of the VRI procedure, although it was shown that a "perceptually-guided cue discrimination" procedure was also effective. The latter procedure, parenthetically, is a "conceptual" prompting method in that configuration of dots representing different numbers (such as on dice) aid in the "discrimination." The verbal-rule instructional method, then, gets strong support from at least three experimental studies.

The most important negative evidence for verbal approaches is presented by Inhelder and Sinclair (1969) who report that providing children with training in the language concepts that are employed by conservers does not lead to successful conservation. It does not appear, however, that a rule instruction procedure was used that parallels the Beilin method. One may conclude from the available evidence that verbal rule instruction leads to improved conservation performance. The mechanism that accounts for its effectiveness is the algorithmic function of language that enables language rules to serve in problem solving even in those instances in which an adequate operative structure is not available to the child. Screening procedures function in an analogous fashion (Beilin [a] 1965).

Multiple training strategies

Some studies reflect the view that any demonstration of conservation acquisition through training would devastate the conceptual apparatus of Piagetian theory, and so some investigators have mixed together every possible technique in an effort to foster the acquisition of logical thinking. There are two difficulties in this. First, the demonstration of concept learning does not necessarily embarrass Piagetian theory. It would, if conservation could be taught to four-year-olds who show no shred of evidence of the capacity to conserve. Otherwise, teaching conservation to five-year-olds is no singular achievement, because it can usually be argued that these children have naturally acquired some constituent knowledge of the phenomenon. Second, the mere demonstration of conservation acquisition through training, in itself, does little to extend knowledge of the mechanisms of thought or the way they develop and function.

The type of study that is minimally required is one that makes it possible to associate experimenter manipulations with specific learning achievements. Some studies, such as Kohnstamm's ([a] 1963, [b] 1967), contain so many experimenter manipulations as to make them of limited value as scientific instruments, although they may be noteworthy as

educational polemics. Some investigators exercise more care in their procedures but also emerge with results that cannot be related to a single experimental manipulation. A multiple method study by Rothenberg and Orost (1969), for example, is based on the teaching of the "component" skills of conservation that have been found effective in prior studies. The techniques used by Wohlwill and Lowe, Gruen, and Wallach are employed in the course of training eight component concepts. Significant improvement in conservation performance is reported, but no indication is given of which components are necessary for conservation (e.g., reversibility, certain verbal terms [more-some], etc.). The authors' comment that another set of component skills might serve equally well further vitiates the value of the approach. Another study (Lasry and Laurendeau 1969) is designed to see if the acquisition of logical operations can be accelerated using an improvement in Kohnstamm's methods. In training conservation of areas they used: direct demonstration, exercises on the compensation of empty spaces (translocation of parts), repetition, and feedback. All subjects (21) in the experimental group acquired conservation of areas ("surfaces") compared with few of the controls. As in the other studies cited here, all of which demonstrate acquisition, it is not possible to identify any procedure as more related to successful performance than any other. Some studies go to great lengths to train for conservation. One study, which provided twelve different lessons to train for conservation of number, involved the "concepts" of (1) one-one correspondence, (2) perceptual rearrangement, (3) as many as, (4) more than, (5) fewer than, (6) addition, and (7) subtraction. The training was successful (Harper and Steffe 1968), but the authors, although they are mathematics educators, make no note of the time and costs involved in training for one kind of conservation, which could be acquired by an older child in a few minutes.

A study that based itself on the thesis that multiple classification, multiple relationship, and reversibility involving multiple relations are necessary for conservation (Sigel, Roeper, and Hooper 1966) undertook with a small number of subjects to test whether training that emphasizes discussion between subject and experimenter could be successful. The results indicate that the experimental subjects profited to some extent from the training. The control group (some of whom were lost by attrition) showed no gains. This study is another example in which it is not possible to identify the elements that account for the outcome. One cannot assume, as these studies imply, that all the elements in the mixture contribute to changes in subject performance. It is quite likely that some elements inhibit acquisition or neutralize the effects of other procedures. These multiple method studies, nevertheless, all report success in the

acquisition of conservation, even though it is not possible to identify a common theoretical or procedural dimension among them.

The phenomenal and the real

Braine and Shanks (1965) propose that the ability to conserve is dependent on the ability to make a distinction between the phenomenal and the real (i.e., between phenomenal size and actual quantity). This notion has led to a number of training studies based on this distinction. Cohen (1967) set up tasks, for example, that ostensibly force the child to use spatial cues where he might use numerical cues. She reports, with what seem very weak criteria for conservation performance, that some children as young as four-and-a-half are able to show improved conservation. Halford (1968) compares conserving and nonconserving subjects, where they are required to recognize equal and unequal quantities in differently shaped containers, when a standard for comparison is available and also when there is none. He finds that conservation performance is accompanied by the ability to recognize compensating changes in two dimensions. Some of these skills, as in the "unknown standard" condition, appear to him to be based on intuitive or perceptual estimation, and are not accounted for by any mechanism suggested by Piaget. He feels, however, that the data also disconfirm the contention of Braine and Shanks (1965) that preconserving *S*s cannot distinguish between real and apparent size. A study by Daehler (1970) concentrates on the investigative activities of children in conservation tasks. He presumed that the real-phenomenal distinction would be related to a child's exploratory activities and hence his conservation performance. A task utilizing illusion-producing materials provided a test of the real-phenomenal distinction. The ability to distinguish real from phenomenal aspects of stimuli was found to increase with age, but was unaffected by training. However, very little conservation was evident in any of the *S*s tested.

On the whole, it seems that the distinction between the phenomenal and the real has not added much to an understanding of conservation. No evidence appears to support the conclusion that the child considers the perceptual aspects of a transformation less or more real than the relevant quantitative attribute. In any case, training for this distinction does not seem to result in any significant gains in performance.

Before considering the newer training studies conducted by the Genevans it would be well to summarize the data so far reviewed. In regard to every training procedure but one, there is evidence for the acquisition of conservation, although contrary data exist to support the opposite conclusion as well. It is nearly impossible to equate studies, since the procedures and test conditions differ so greatly, which hardly

ever comes as a surprise in a review of research. The reports of training conditions are so varied and provide so little detail that even the most specific offer insufficient information to determine whether serious confounding of experimental variables has occurred. What one experimenter may consider trivial, such as the nature of what is said by way of explanation to a subject, turns out to be quite critical in the eyes of another. Nevertheless, the striking fact is that almost every training method, at least in the hands of its proponents, leads to improvement in conservation. This fact is of theoretical significance. It is significant also that methods other than those directly relevant to the Piagetian theory are as successful as they are, often more successful than Piagetian methods. A fact of equal significance is that no method seems to be effective with very young children (i.e., four-year-olds—those truly preoperative). The significance of these generalizations will be discussed later.

Recent Genevan training research

The recent Genevan training studies, which represent a third wave of training studies, are by intent a departure from the earlier Genevan studies. Their stated aim (Inhelder and Sinclair 1969; Inhelder, Bovet, and Sinclair 1967) is to study the psychological mechanisms that underlie the transitions between stages. They address themselves to at least three questions.

1. Is the pace or rate of cognitive development under the control of a mechanism analogous to that which functions in embryology? To answer this question they studied preoperational, intermediate stage, and operational children in a conservation of continuous quantity (liquid) task. The apparatus was so constructed that a child could make a conservation prediction and confirm his prediction immediately. An intensive dialogue between experimenter and child accompanied the manipulation of materials. No preoperational subject learned to conserve, and about 12 percent reached an intermediate level of operativity. Of the subjects who started at an intermediate level, 25 percent made no progress. For 38 percent, there was slight progress that represented an extension of thoughts they already had, but with 37 percent of the subjects there was a "true" elaboration of their reasoning to true conservation. In this training condition there were many explanations with appeals to identity and compensation, but few to reversibility by inversion ("annulment") that is common with natural conservers. An operative level group (i.e., those who had conservation of number), were trained for conservation of weight, which is acquired normally two or three years later. Only 14 percent made no progress. In addition, those who acquired weight con-

servation were also capable of performing operations of transitivity. Inhelder and Sinclair concluded, however, that operational children who learn in experimental conditions have a quality of response very different from preoperational children who are trained. On the basis of these findings it is concluded that the acquisition of conservation is under the control of the processes of development.

A non-Genevan study that also concerned itself with developmental pacing (Beilin and Franklin 1962) showed that in training six- and eight-year-old children in length and area qualitative measurement, older Ss could be taught to do both types of task, but younger Ss could make progress only in acquiring length measurement.

2. Another question concerns the developmental links between partial structures: Can the acquisition of elementary spatial measurement (of length) be facilitated by the child's application of numerical operations? If yes, what stages does the child go through? Further, does the child's behavior during learning sessions provide insight into the interval that elapses between acquisition of number and acquisition of length conservation? Inhelder and Sinclair report a study in which children who had number but not length conservation were trained in a length conservation task. They were required to construct a straight line equal in length to a zig-zag the experimenter had laid out, in one case with matches equal in length to the standard, and in another with smaller matches than the standard. Extensive discussion similar to a clinical interview occurred between the experimenter and the child. About 35 percent of the subjects made no progress at all, but of the rest 28 percent gave correct answers and full justification. The nature of the child's performance led Inhelder and Sinclair to conclude that it is possible to use already acquired numerical operations to lead to the operations of spatial measurement; however, progress is very slow if one wishes to reach full acquisition. The causal mechanism of change in these experiments is held to be the conflict resulting from the misinterpretation of misleading cues, a conflict which is overcome only by a constructive effort to discover "compensatory and coordinating actions." It is feedback from the subject's actions and not from the visual results of the experiment that leads to structures of a higher order.

3. The last concern of the Genevans is with the role of language in the acquisition of operative structures. They characterize the prevailing ideas of the relation between language and thought (such as Bruner's "instrumental functionalism") as holding to the view that in the course of development the child fits his experience to his language, and that the verbal representation of objects and events permits the child to acquire concepts and operations.

Inhelder and Sinclair report a series of experiments in which the features that distinguish the operational from the preoperational child are not the lexical terms used by the child, but structural devices, comparative markers, and sentence structure associated with the child's thinking. With this knowledge as a base they attempted to teach preoperational children the patterns of expression used by operational children. The training showed that it is easier to teach preoperational subjects the correct use of pairs of verbal opposites than to make them use comparatives and coordinated sentence st: tures typical of conservers. Even subjects who did acquire superior verbal forms rarely progressed to operativity (only 5 percent conserved). Verbal training, however, did lead children to attend to various dimensions and their covariation but not to a sufficient extent to arrive at a multiplication of relations or to relate them to a reversible transformation. In a seriation-of-lengths experiment, only 30 percent could describe a stick as bigger than another and at the same time smaller than a third (i.e., acquire transitivity) as a result of linguistic training. Inhelder and Sinclair conclude from these experiments that linguistic structures are not acquired uniquely according to their own laws. On the contrary, an operative component is necessary before linguistic structures can be generalized.

In sum, what these newer studies lead the Genevans to conclude is that the development of operativity is malleable only within the limits imposed by the nature of development. They assert quite vigorously that preoperative children do not acquire true operativity even with training. Although learning may accelerate development, acceleration is limited by the conditions of assimilation and children assimilate less of this learning in earlier stages of development. Although the possession of an elementary invariant (e.g., conservation of number) is a prerequisite to success at more advanced operative stages, the possession of a structure in one field does not lead easily to the acquisition of another. In fact, it may not lead to progress in another field at all. Further, language is but an instrument, and learning capacity is not provided by that instrument. Learning then, is subordinate to the laws of development, which itself follows laws that are both logical and biological.

It can be seen from this exposition that in one sense the Piagetians have moved a distance from their prior positions, and in another way they have only reinforced their earlier views. In a 1959 statement regarding training studies, Piaget made a distinction between physical and logico-mathematical experience. He emphasized that training based on physical experience could lead to successful conservation performance in those cases where the type of conservation involved physical elements. This explained why Smedslund was successful in training weight conservation.

Logicomathematical experience, however, was either reduced to physical experience, and training in this case was not likely to work, or it became an operatory experience, in which case it could speed up acquisition. While the distinction between physical and logicomathematical experience is still used, the distinction is blunted by the recognition that neither experience is independent of the other. What is retained from the earlier position is the notion that training is successful because it attaches the operations to be taught to ontogenetically prior operations that are themselves preparatory to the development of more complex structures. The conviction that learning only accelerates development but does not initiate it, and is thus under the control of developmental processes, is a reiteration of long-held views. A new element appears in the recognition that starting with a structure in one domain one can through training lead to acquisitions in related though more complex domains, although the process is long and difficult. Why there should be so much difficulty in the generalization of operations that are part of a larger common system (i.e., concrete operations) is still not clear. Why the conservations do not generalize more easily when they differ principally in conceptual properties (e.g., length or number) is puzzling, particularly when Piaget emphasizes that the more difficult acquisitions are those involving the operations themselves.

To the Genevans, then, training studies are still seen as means of showing how development can be accelerated, not initiated. The important tasks ahead for the Genevans are not those of defining the nature of the mechanisms in conservation, since these are apparently known, but rather to discover more of the functional relations among the operations.

Training of Logical Operations

Most of the training studies have been concerned with conservation. Those that go beyond conservation are concentrated in a limited number of areas, although the possibilities for training extend much more broadly. One would surmise that the principal theoretical and practical issues in which investigators are presently interested can be tested and explored with a limited number of logical operations.

Class inclusion

Class-inclusion problems have received a fair amount of attention. Morf (1959), in the first wave of Genevan training studies, devised three training techniques to determine whether logical structures were the result of direct (external) experience or internal equilibration. Subjects were

selected who understood and could isolate classes but could not understand the class-inclusion question, particularly as it used to be asked by Piaget: "Are there more flowers than roses?" Training by demonstration led to little improvement. Subjects trained with free exploration, when the exploration had something to do with the problem, showed definite improvement, but Morf could not decide whether the change was a true change or resulted because subjects were on the brink of operative reasoning. Training that involved concretizing nesting schema by giving the child perceptual clues seemed to be of little value, whereas training in "multiplicative relations"—that is, with the intersection of classes, double functions of a single object, and double dimensions in intersection or nesting—was most effective. Morf observed that with free spontaneous classification the only cases of operative success were those that came at the end of the activity, which suggested to him that the new achievements resulted from the consolidation of already present schemes. Piaget interprets Morf's study (and the studies of Greco and Smedslund) as showing that in order to construct and master a logical structure, the subject must start from another, more elementary logical structure, which he will differentiate and complete.

As previously indicated, Kohnstamm ([a] 1963, [b] 1967) was not content with the Genevan conclusion that operativity could not be achieved by class-inclusion training. The Kohnstamm training approach was to provide a total educational experience, which included the pointing out of wrong answers, explaining how the answers were incorrect, the use of leading questions, and the use of both verbal and nonverbal materials. Kohnstamm was able to demonstrate effective learning by the use of his total procedure. As indicated before, it is not possible to determine which elements in the training led to the results. Kohnstamm considers this total attack to be a virtue because of its flexibility and because it most aptly approximates what is done in educational settings. At the same time, the Piagetian method, which is equally flexible, is attacked because of the Genevan unwillingness to provide the child with problem solutions in verbal form. On inspection, Kohnstamm's method seems to reduce to a verbal rule instructional procedure with flexible conditions of administration. There have been three Piagetian responses to Kohnstamm's study (Pascual-Léone and Bovet 1966; Lasry and Laurendeau 1969; Inhelder and Sinclair 1969). Pascual-Léone and Bovet (1966) criticize Kohnstamm because he relies on what they interpret to be "figural structures" that do not necessarily correspond to the underlying logical structures, because the instructional procedures are didactic, and because his method relies on building up a set of capacities to neutralize contrary cues through discrimination learning. They think solutions are achieved

through the acquisition of "empirical schema" (i.e., empirical generalization) rather than operative structures. They also criticize the experiment because the same materials are used in training and in the posttests. Lasry and Laurendeau (1969) replicated Kohnstamm's experiment with some refinement, although they use the same multifactor design. Particular emphasis is placed on determining whether the acquisitions are truly operative by examining subjects' justification of response, as well as through the use of a transfer design and a control group who are at an operative level. Like Kohnstamm, they obtain a significant amount of learning. Subjects were more successful with material class-inclusion problems than with verbally presented problems. They also had much less success with the verbal problems than Kohnstamm (54 percent compared with 83 percent), which may be because their criteria are more stringent than Kohnstamm's. They believe there is evidence that operativity was achieved but this is qualified by the observation that available criteria for operativity are too vaguely defined to make it possible to judge such achievement. The response of Inhelder and Sinclair (1969) to the Kohnstamm study is unequivocal. They do not dispute Kohnstamm's results, but they do dispute the interpretation and conclusion that operativity has been achieved. To demonstrate this they undertook a learning experiment in class inclusion with eleven children. When they use Kohnstamm's criteria for judging achievement, nine of the eleven succeed. If they continue to add what they consider to be different criteria for operativity, they find a perfect Guttman type of scale reflecting the following ascending order of criterion difficulty: (1) simple class inclusion, (2) three-stage class inclusion, and (3) valid explanation and correct response to a problem demanding an answer of a different form. With the most difficult criterion, only two of the eleven succeed. They therefore still question whether an operative structure can be acquired by "empirical-didactive" methods.

Another class-inclusion study (Iijima 1966) employed labeling as well as exercises in addition/subtraction and one-one correspondence to foster knowledge of class inclusion. There was no learning with four-year-olds with either method, but there was acquisition using both methods with older children. The author notes that the number of elements that have to be counted affects the child's ability to relate classes. A related observation by Wohlwill ([b] 1968) is that there is a tendency to incorrectly reduce the total class to majority class comparison to a comparison between the two subclasses (i.e., the larger vs. smaller class). Wohlwill's study dealt with the effect of practice in abstracting class and ordinal relations upon the ability to deal with class-inclusion problems with both pictorial and verbal materials. He concludes that performance in the

class-inclusion problem is affected by perceptual sets, that is, the perception of two contrasting subsets unbalanced as to number that creates a strong tendency to make the problem one of subclass comparison. He comments further that counting and practice should lead to improved performance. Although these factors are important, Wohlwill considers them subsidiary to the more intrinsic factors related to cognitive development. Ahr and Youniss (1970) pursued these issues further with a training study in which there were two conditions. First, there was "expanded-question training" in which the subject was asked questions in the form "Are there more pets or more dogs or more cats?" which was designed to overcome the tendency to substitute a subclass for the superordinate class. The second type of training involved a feedback procedure with correction for incorrect responses. Six- and eight-year-olds were trained. The feedback training with correction proved more effective, and older children benefited more than younger. The authors conclude that the correction procedure does not promote inclusion behavior de novo but "serves to bridge a gap between not yet stable comprehension and performance."

These studies of class inclusion point to the fact that training can lead to successful acquisition of this logical ability. The methods of training that have been successful have been verbal with some kind of rule instruction procedure used in most instances.

The question of operative achievement from instruction and training still appears not fully resolved. The issue as to how operativity is to be both conceptually and operationally defined remains. As long as there are conceptual differences among investigators there will probably be differences both in the conceptual definitions and in the criteria that are accepted for determining the existence of the operations.

Classification and relational thinking

There are few data bearing on the training of classification and relational skills except for the class-inclusion studies just cited. While there is considerable research on rule learning with classification concepts in the general experimental literature, it has little direct bearing on the issues raised by Piagetian theory.

One relevant study (Hooper 1970) reports on the effects of label classification training, seriation training, and memory-discrimination training on the acquisition of these skills in addition to their transfer to conservation in a group of young children approximately four and five years old. Seriation training was superior to classification training, but neither had any effect on conservation, contrary to data reported earlier by Sigel, Roeper, and Hooper (1966) for an older group of subjects.

Studies of conservation, particularly conservation of weight, appear to some investigators to involve the operative capacity of transitivity. A study by Garcez (1969), which attempted to determine the effect of empirical demonstrations with a scale on both conservation and transitivity of weight, reports that none of those trained on transitivity alone became operational on the posttest, but 24 percent of preoperative Ss trained on both transitivity and conservation gave operational responses. No control Ss were operational, but 28 percent of those given only conservation training became operational. With transitional subjects there was even greater acquisition of transitivity, particularly among those with conservation. Training conservation alone did not lead to greater transitivity success, however. The author concludes that operational achievement of conservation is needed for transitivity and that without operativity empirical demonstrations are ineffective. A study by Smedslund ([c] 1963) was designed to test the effects of empirical control vs. no empirical control, and fixed vs. free procedures on the acquisition of transitivity of weight. Although about 30 percent of the subjects acquired transitivity, it could not be assigned to any particular experimental procedure.

An extensive study of the effects of school instruction on the acquisition of logical operations was conducted by Almy and her associates (1970). The intent was to determine whether instruction in the "new" math and science programs in kindergarten and first grade had led to more advanced levels of thinking in the second grade, compared with the achievements of children who had not received such instruction. Tasks used as criteria of achievement were number conservation, class inclusion, seriation, transitivity, classification matrices, and conservation of weight. The results which reflected on the efficacy of the special programs were disappointing, although those who had early instruction of any kind were on the whole better than those who did not. Participation in both mathematics and science programs seemed to produce performance superior to participation in only the mathematics program, although the latter group did better on the serial ordering tasks. The authors concluded that on the whole the effects of school instruction in the mathematics and science program did little to foster the integration of logical capacities at the second grade.

A member of the Geneva group (Gréco 1959) reports a study in which children were trained to deal with inversions of serial order through 180° transformations. His intent was to compare experimentally trained with naturally acquired logical structures. Two different "clinical method" training situations were employed. In one, only a single training session was used with a repetition of items. The other situation was one in which a succession of items was given over a period of several sessions. Only the latter approach was successful. Gréco sees two outcomes from training,

one in which empirical observation leads to unstable understanding, and the more successful method which leads to stable understanding and "structuration" more amenable to generalization. He concludes that training can lead to acquisitions that are equivalent to natural acquisitions, although trained learning appears to be less stable than operations acquired naturally a year or two later.

A study by Ojemann and Pritchett (1963) makes use of demonstrations given over a period of three sessions to train the concept of specific gravity. The demonstrations were with lighter and heavier objects, which were floated and weighed. The experimental group, at both kindergarten and first-grade level, surpassed the controls, although they were at a higher I.Q. level to start with.

As with conservation, Soviet investigators have undertaken a series of studies for the training of the logical structures related to classification. The experimental procedures are based on an analysis of the components necessary to the correct solution of class-membership and class-relations problems. The experiments involved training in (1) the differentiation and identification of object features, (2) learning the actions involved in class membership, and (3) learning to distribute objects into hierarchically connected classes (class-inclusion relations). The data, which are not reported in detail, are said to show that six- and seven-year-old children are able to learn these classification relations, which for Piaget's samples occur a number of years later (Teplenkaya 1966).

Two further studies show the successful effects of training (Beilin [a] 1965; Beilin, Kagan, and Rabinowitz 1966). In the former instance it was shown that a learning effect was evident only with older trained children and not the younger. The older children learned to measure both lengths and areas, whereas the younger children could only learn to measure lengths. The latter study demonstrated that the confirmation of predictions of shielded water levels by a confrontation with the visual reality led to more successful performance than training based on a verbal instructional method. It was concluded that the type of concept determines its susceptibility to certain types of training. While verbal rule instruction had been shown to be successful with conservation (Beilin [a] 1965), it was not so with water levels. The visual display, in this instance, provides the data out of which the logical structures were built.

The review of research on the training of concrete logical operations leads to the conclusion that these operations are as subject to the effects of instruction as are those of conservation. The problems are also the same as those of conservation. A variety of training methods lead to operational attainment. Whether the attainments are truly operational is again open to interpretation and may be a function of the criteria

used to judge the response of subjects to particular experimental manipulations.

Formal operations

Considering the significance of Piaget's contention that logical thought, reflected in the use of the propositional logic, becomes possible only in preadolescence, it is surprising how little research is addressed to this issue. The most important general property of thought of this period (Flavell 1963) concerns understanding the relation between the real and the possible. During this period strategies based on the reasoned determination of all logical possibilities are adopted by the problem solver. These strategies are then related to an assessment of the activities one is able to carry out. The final result is the construction of a causal structure to account for the data of the problem. This sequence is particularly manifest in problems requiring combinatorial thinking. A study that is ostensibly designed to test for some of the properties associated with this period is one by Ennis and others (1969), who trained first-to-third-grade children in "conditional logic." The training was of five logical principles: (1) basic understanding of the proposition, (2) contraposition, (3) conversion, (4) inversion, and (5) transitivity. In addition to determining whether these principles could be taught by the instructional method proposed by Ennis, an attempt was made to test the Piagetian claim that young children do not reason from premises without believing in them. The experimental procedures involved the use of a toy house with a handle on it. The house was used to represent a set of possible conditional relationships; for example, "If the handle is up, the bell does work." Teaching the logic was accomplished with tape-recorded instructions in a booth containing a variety of instructional materials and objects. Fifteen weekly programs were used. It is reported that teaching resulted in no differences between the experimental group and the control. It was found, however, that many subjects had already mastered the basic principles (ranging from 13 percent to 45 percent on the various types of task), which led the author to claim that conditional logic can be mastered by children younger than eleven or twelve years of age, contrary to Piaget's assertions. However, the Genevans themselves make the point that the concrete-operational child (age seven to twelve) has the capacity to deal with inversion and transitivity but lacks the ability to coordinate or group these elements into a structure that can be used in problem solving. The critical question is whether children can use and coordinate conditional statements in a situation in which the deductive and actual possibilities become known to the child, and where he is able to carry out this construction without the necessity of a trial-and-error

manipulation of materials. These are not, apparently, the conditions of the Ennis experiment. It is also reported by Ennis that factual items are better handled than suppositional items, which is consistent with Piaget's assertions: but since the absolute number of supposible items is high, it leads Ennis to question Piaget's claim. Again, since the main part of the study is not a clear analogue to a Piagetian experiment, it is difficult to interpret Ennis's claim. This is another instance in which an apparent difference in conceptual definition makes interpretation of the results a matter of dispute. By Piaget's criteria, these results are perfectly understandable and interpretable. In Ennis's terms, Piaget's theory is weakened by the data. A study by O'Brien and Shapiro (1968) of children's ability to deal with "hypothetical reasoning" comes to a conclusion quite different from that of Ennis. It was found that in a sample of nine-to-seventeen-year-olds only 14 percent of the fourth-graders' responses and only 45 percent of the twelfth-graders' were correct in interpreting statements deductively arrived at from if-then propositions, a conclusion at variance with the Ennis claim. Another effort to train preoperational children in a complex problem-solving skill shows that some skills can be learned, but where the problems are close approximations to the type of task Piaget uses (e.g., combinatorial tasks), training efforts are not very effective (Anderson 1965).

A series of studies by Smock and his students (e.g., Leskow and Smock [1970]) involve training in permutation problems. They find a steady increase between twelve and twenty years in the use of problem-solving strategies that reflect the properties of a mathematical group, as well as in the specific strategy of holding the initial elements constant while rearranging the other elements. In training to solve these permutation problems, however, a practice with feedback procedure was not effective in producing significant gains.

Fischbein, Pampu, and Mânzat (1970) used three instructional procedures in training probability concepts with preschoolers and third and sixth graders. One of the procedures appears to be verbal rule instruction of a grouping principle. Another training procedure involved the illustration of the grouping principle. These techniques were successful in promoting the ability of third and sixth graders to perform chance estimates by comparing numerical ratios. The authors question Piaget and Inhelder's conclusion that the ability to deal with proportionality is accessible only at the formal-operational stage. They also question the "law of stages" by virtue of the demonstrated effects of training. Although the study provides interesting data concerning subject strategy shifts by virtue of training, the results would be more convincing if a transfer design had been used.

Experience and logical operations

The intent of many training studies is to identify the mechanisms in conservation that are acquired in the course of natural development. They aim to expose the nature of cognitive processes and structures and at the same time characterize the elements of experience that induce or have a bearing upon cognitive change. It is interesting in this light to pursue what is known about the effects of natural experience upon conservation and other types of logical thinking. There are two kinds of information available. One source is the group of "natural conservers" identified in conservation experiments. They are not as a rule the subjects of study but are set aside in favor of those who fail to meet the criteria for conservation. Natural conservers occasionally are used as contrast or control groups. In one study (Beilin [a] 1965) the generalization effects of conservation are contrasted between natural and trained conservers. Generalization was found to be greater among the natural conservers than among the trained subjects. (That is, of the natural conservers who passed two of three conservation tasks, 60 percent also passed the third. Of the trained conservers who reached criterion on the two trained tasks, less than 30 percent could succeed in the third task.) A second source of information about natural conservers comes from their reported ability to resist the extinction of conservation upon the presentation of contrary evidence (Smedslund [b: III] 1961). Because of this ability the counter-suggestion extinction technique is widely used as a criterion to establish operative conservation in subjects. Recent evidence shows, however, that even older natural conservers (college students) can yield to the extinction procedures (Hall and Kingsley 1968). Given the proper circumstances, then, conservers can be made to appear like nonconservers even in those circumstances when this defies knowledge of their true state.

Another type of study has been used to detail the effects of experience. This is the quasi-naturalistic, cross-cultural, and subcultural comparison method. The technique has been used to determine (1) whether children in different cultures acquire logical operations at the same time, (2) whether they acquire them in the same order, (3) whether the cognitive mechanisms are the same in all cultures, and (4) the differential effects of experience upon the construction of operations.

1. *Acquisition time of logical operations across and within cultures*

There seems to be general agreement that the age of acquisition of logical operations differs as a function of cultural experience. There are differences reported between cultures (e.g., Greenfield 1966; Maccoby and Modiano 1966; Goodnow 1962; Goodnow and Bethon 1966)

and within cultures (e.g., Almy, Chittenden, and Miller 1966; Almy et al. 1970; Beilin [a] 1965; Beilin, Kagan, and Rabinowitz 1966; Price-Williams, Gordon, and Ramirez 1969). These studies show that Western societies tend to accelerate acquisition (e.g., of conservation), although some argue that the differences are due principally to schooling. Subcultural differences among social classes are also reported. There are some studies, however, that show no differences that are attributable to schooling (e.g., Mermelstein and Shulman 1967). One set of studies shows no cross-cultural differences (or differences due to schooling) in conservation performance but does show it for combinatorial thinking (Goodnow 1962; Goodnow and Bethon 1966).

2. *The order of acquisition*

None of the cross-cultural or subcultural studies shows an acquisition order that is essentially different from that reported in Swiss, American, or European studies.

3. *Consistency in mechanisms*

Some cross-cultural studies focus specifically on the nature of the mechanisms involved in logical thought (e.g., Greenfield 1966; Maccoby and Modiano 1966; Voyat 1970). The Harvard studies of conservation and classification behavior in Africa and Central America employ the same categories of analysis for all cultural groups, which suggests that the nature of the logical mechanisms is the same even though the data show that cultural experience results in differing emphases in the use of cognitive strategies. The results thus support a weak Whorfian thesis. Voyat's data (1970) show that Sioux Indian children are attaining the same types of cognitive skills as Genevans, although they may not accord precisely in their rates of achievement.

4. *Differential effects of experience upon logical thought processes*

Identification of the parameters of cultural experience that affect the rate of acquisition and the differential use of cognitive strategies has been approached by only a few researchers. Greenfield (1966), Maccoby and Modiano (1966), and Bruner and others (1966) emphasize the effects of schooling and the distinctive differences between rural and urban environments. The lack of schooling appears to place a ceiling on conservation (Greenfield 1966), although Piaget (at the seminar held at Catholic University 6 June 1970) has expressed the opinion that in their work activities members of aboriginal cultures probably express higher (i.e., formal operational) levels of thought. Greenfield proposes that schooling creates an analytic orientation to perceptual processing that permits higher performance levels, which

seems consistent with the argument that successful conservation requires an analytic-set in order for conservation to occur (Beilin [c] 1969). Price-Williams, Gordon, and Ramirez (1969) demonstrate better conservation performance (particularly in a rural area) among potters' children who help in the pottery than among children of the same social level whose parents are in other occupations. They propose that skill learning by bringing the cognitive mechanisms in coordination with objects leads to operative development. Another argument frequently heard (e.g., Maccoby and Modiano 1966) is that Western industrialized society forces an abstract attitude on children, whereas rural children (e.g., Mexican Mestizo) learn to make finer perceptual distinctions.

As cultural dynamics these represent abstract causal factors that are embedded in behaviors, attitudes, and social customs. An analytic set, an abstract attitude, a perceptual orientation, and so on, that differentiate urban from nonurban cultural contexts are quite abstract properties of performance. How they are actually mediated so that they affect differential cognitive acquisitions is not clear. Whether such abstract notions act as the true parameters of experience that create cognitive change is far from substantiated. What is clear, though, is that these experiential variables do not create new or different logical thought mechanisms. They trigger their operation, they accelerate their development, or their deficiency inhibits development.

General Discussion

The considerable number of studies that concern themselves with the training of logical and infralogical operations attests to the importance developmental psychologists and educators associate with the issues treated in them. It also attests to the power of the Piagetian claims that so many investigators desire to illuminate, refute, or support them.

The data from these diverse studies show that training makes possible an improvement in performance in practically every type of logical or infralogical operation. While the Genevans might concur in this generalization, they would not concur in an assertion that true operativity is achieved by virtue of training where no vestige of operativity existed before. There are two difficulties in determining whether true operativity is achieved. First, there is no agreement as to the nature of operativity. Non-Piagetians are not convinced of the need or efficacy of the concept of operativity, preferring other conceptions. Behavioristically oriented psychologists in particular reduce "operations" to processes such as atten-

tion, set, or various learning parameters more consistent with learning or behavior theories than cognitive theories. Those with cognitive orientations may prefer other functional or structural explanations. Since such theoretical differences lead to differences in the interpretation of data, there is no easy resolution of the conflict over whether operativity is attained, particularly if there is no agreement about the very existence of operativity.

There is an additional difficulty in the measurement definition of operativity. Disagreements about the nature of operativity make it difficult to agree on the use of such measurement criteria as resistance to extinction, transfer, verbal representations of strategies, time delay, and nonverbal performance criteria. It may be that a hierarchy of criteria is needed, as suggested by the Inhelder and Sinclair (1969) work, in which levels of operativity can be conceived in relation to weak and strong criteria. The Genevans insist on strong criteria because the architecture of the theory seems to depend on them. The use of other, at best, weaker criteria makes it easier to disconfirm Piagetian claims. Much needless controversy is undoubtedly created by differences as to what constitute acceptable standards of analysis.

The extensive evidence of performance gains from training transcends the problem of conceptual and operational criteria of operativity. What emerges from the data is the striking fact that a wide variety of methods—in fact, practically all types of experimental method—lead to successful improvement in performance, even if not in every experiment. Here the Genevans appear on their weakest ground. Even if their thesis is correct that new structures are only constructed out of the confrontation of different or contradictory schemas, they make no effort to define the dimensions of experience that lead to this conflict, except to imply that use of the clinical method will accomplish it. Even so an apparent contradiction exists because the clinical method is extensively verbal and the Genevans have been at great pains to insist that verbal methods will not lead to operative achievement. While we may agree that the child's logical capacities reflect the construction of operations and structures out of experience, it is more likely that the elements of experience out of which schemas can be constructed are of such a variety as to include verbal experience. They may not include every conceivable experience, since some criterion of relevance is necessary, but rule-embodying experiences certainly seem to represent a source from which schemes may be constructed even if they may not invariably result in true operations. The Piagetian clinical interview is itself a source of some ambiguity, however, since it has not been carefully analyzed in relation to the character of the information conveyed to the child through the form of the questions asked,

their order, the vocabulary content of the messages, the rule-ordered properties of the organization of inquiry as well as the organization represented in the materials themselves. I have previously suggested (Beilin [c] 1969) that the actual transformations that occur in a conservation experiment, in addition to creating the problem that the child has to solve, also provide the relevant information to solve it. How extensively this phenomenon is embodied in the clinical method is not known. Nonclinical training methods, including those used by behavioristically oriented researchers, may in fact encapsulate in a very refined fashion some of the elements of the Piagetian method.

The data demonstrating acquisition with a wide variety of training methods give further support to the idea that the logical operational system is under the control of a genetic mechanism that only permits the programmed development of defined cognitive structures through interaction with environmental inputs. The environmental inputs may be quite varied in this circumstance, since the control of the developed structures comes from internal programmation rather than from the regularities of the environment or from regulated experience with the environment. Since the child is an active agent in the acquisition process, he can construct a conceptual system out of a great many materials and techniques, even those unintended by the researcher. The very fact that the child is able to construct schemes out of these diverse materials and experiences shows not only that the child is active but that the basic pattern of organization is internal. The relatively wide diversity of inputs that lead to the acquisition of conservation and other logical thought processes argues, then, for the fact that internal programming acts strongly and selectively to control the development of these capacities. It argues also for the use of nonexclusive methods in the instruction and training of these logical processes, although some methods and contents may be more relevant than others. While some methods may be quite specific, others may be so diffuse, general, and abstract as to require a complete instructional system in order to create an effect. The analytic set or abstract attitude which appears to develop in urban industrialized cultures may be acquired through a generalized instructional attitude that orients teachers, parents, workers, and scholars, and ultimately children themselves to respond to their world with a disposition to analysis and idea construction. This analytic set may not be formally taught but is a concomitant of the instructional style common to the broad class of formal and informal acculturation agents (i.e., parents, teachers, etc.).

Piaget, in response to requests to define the relevance of his developmental psychology for education, usually responds (e.g., at the Catholic University seminar) by questioning the aims of education. He says that

one may wish an educated citizenry who merely receive and transmit the knowledge of the culture, or one may wish the members of a society to be creators of knowledge who respond constructively to their experience. If the latter is the case, then another alternative in education is required. Piaget obviously opts for and recommends the latter course of action, with the implication that an educational approach that embodies the elements and principles of the clinical method would serve this objective, although Piaget and the Genevans have been loath to specify what an educational curriculum should be either programmatically or in detail (Constance Kamii is a notable exception). There are three difficulties with the Piagetian thesis, even if one concurs in his social objective. First, it is never clear in the constructivist approach to knowledge acquisition as to the nature and extent of required prior knowledge that can only be acquired by "rote" or didactic methods. How much memorization of the number system (i.e., learning the sequence, the symbols, etc.) is required, for example, in order that a child be able to think constructively in regard to number relations or classifications? In Piagetian experiments these aspects are usually taken for granted. They are not irrelevant, however, to a consideration of what is required in an educational curriculum. Second, and more important, is a required recognition of the state of present-day education. Education everywhere in the world, except for a very privileged minority, is group education. It is becoming increasingly organized in terms of a mass technology, with more children per teacher, more materials and more instrumentation per class. The time spent by a teacher with an individual child is constantly diminishing. Individualizing instruction is increasingly a myth, and instrumented individualization is also a myth, as has been pointed out by Piagetians themselves. The Piagetian method, on the other hand, places its primary emphasis on one-to-one teacher-student interaction with individualized teacher response. This is economically and tactically impossible in today's increasingly crowded schools. What is needed, on the other hand, is vigorous pursuit of how "constructive" educational approaches can be realized with groups of children, which would require instructional strategies different from the one-to-one "clinical method." Thirdly, Piaget's skepticism of verbal instructional approaches seems limiting for both theoretical and practical reasons. The relationship between language and cognition is not very well understood. Piaget's view that language develops as a system for the representation of thought, although it has evidence to support it (e.g., Inhelder and Sinclair 1969; Beilin [c] 1969), is far from explaining the full nature of the relationship! It is evident that language data are utilized in the construction of thought, even if for the young child they do not have the same salience as sensory and per-

ceptual experience. It is important that this be recognized, since many of the significant cognitive experiences of the child are mediated through language, particularly where interactions between parent and child are concerned. As Piaget himself makes clear, while action is the basis for cognitive attainments, the critical actions are not physical but implicit mental actions, that is, mental surrogates for actions such as are represented in mental transformations. These "actions" can occur in linguistic as well as nonlinguistic contexts. The evidence that reversibility transformations occur in the syntax of natural languages (e.g., in the active-passive transformation) supports this view. The question of how linguistic information that schematizes the operations of thought can be constructed into knowledge is not addressed by the Piagetians. Instead, verbal data are interpreted as concentrating attention on various features of visual displays or on the transformation of such displays. Even when verbal concepts are studied, as in whole-part classification experiments, the Genevan emphasis is still on nonlinguistic phenomena, although the phenomena are linguistic in significant respects. Although Piaget is correct in emphasizing that preschool and early school education often concentrate too much on language learning, more information is needed of the extent to which language provides the necessary architecture for the acquisition of knowledge. What would early Piagetian acquisitions be like, in fact, without the language context in which they are usually acquired. A more desirable stance to take vis-a-vis language in education, then, is one aimed at discovering how it can help in the construction of knowledge. What the studies here reviewed suggest is that the linguistic interchange between experimenter and child is a significant feature of all training methods—including the Piagetian and behavioristic methods, even when their focus is not on linguistic contents. It is unrealistic to believe that the linguistic communication system is completely neutral relative to the contents communicated. Piaget's point that language teaching is not sufficient to the acquisition of logical reasoning in children, then, neglects the prospect that language data provide elements that are capable of being constructed into knowledge and that language is itself an activity that embodies operations.

The training studies discussed here can be meaningfully interpreted in terms of the competence-performance distinction proposed by Chomsky in respect to language acquisition. Many of the studies that are critical of Piaget's cognitive theory are in effect proposing explanations of conservation and the logical operations that are more in accord with a performance model than a competence model (Flavell and Wohlwill 1969). The phenomena of attention, learning sets, analytic sets, and others represent psychological properties that affect the expression of basic

competence and do not touch on the competencies themselves. The competencies are more appropriately represented by the equilibration mechanisms that Piaget is positing as fundamental to the development of cognition. Although such factors as attention and various features of learning are intimately related to the child's ability to perform and to the realization of basic competencies, they do not directly relate to the formation of the competencies. While these experience-bound parameters have a great deal to do with the development of strategies by which cognitive structures are made functional, at this point little is known as to how this occurs.

Lovell (1966), in addressing himself to the implications of Piagetian research for mathematics education, pointed to both the assets and the limitations of the Piagetian theory. By way of summary, we might add to Lovell's catalogue the following:

1. While Piagetian research has shown how complex a process the growth of mathematical concepts can be, it has not demonstrated how it can be made easier. Although Piaget describes the processes by which thinking develops as a series of constructions, he does not suggest an educational technology by which such constructions can be made to occur. The direct translation of the clinical method, which is useful as a technique in the discovery of the constructive processes, to a technology of educational instruction is at this stage unwarranted. While the technology for achieving change has to be related to the mechanisms of change, it does not require that the constructive mechanisms themselves serve as the technological model.

2. The idea that the child has to be active in contrived situations involving conflict to acquire logical reasoning is not substantiated by the available research. Neither active problem-creating conditions nor conflict-creating situations are necessary for logical thinking to be acquired. A wide variety of techniques and contextual conditions contribute to or, at the least, permit concept acquisition. Nevertheless, as the data show, there is a limit placed on these acquisitions by the developmental level of the child. No logical or mathematical learning is likely to occur, at least without great difficulty and tenuousness, if the concepts to be learned are far beyond the operational level of the child's available cognitions.

3. While fairly clear divisions can be made between gross levels of mathematical abstraction, it is not clear that even gross hierarchic systems can be easily established on an a priori or even empirical basis. Differences of opinion inevitably occur as to the nature of conceptual or operational hierarchies. Even an a priori system that is empirically

tested will encounter differences in interpretation—nevertheless, this is still the best method available for determining the order in which mathematical concepts should be taught. Using empirical as well as logical methods is the only reliable way to establish vertical hierarchies, branching hierarchies, or other complex relationships among concepts and conceptual systems. Mathematicians who choose to teach a sequence of mathematical concepts and functions on purely a priori bases may encounter great difficulty in having these concepts learned. Logical relations are not inevitably paralleled by psychological relations. Unfortunately, little effort has been expended in testing the relations between the conceptual systems of mathematics and the cognitive system of the child except in the most limited of circumstances.

4. While Piaget has noted that the cultural context can inhibit or facilitate learning, he does not define the conditions that specifically determine these effects. In some instances, he suggests, the technological demands of a culture create pressures for the attainment of highest levels of operational thought. Little else is done, however, to suggest how cultural experience creates cognitive change (Sigel 1968).

5. Transfer of training has been a challenge to both psychology and education. The limits of transfer through training have been apparent in a long history of studies, and the Piagetian studies are no exception. Piagetian theory has been interpreted by many to imply that transfer should be easy by the very notion of common schemes or operations functioning within a developmental level. Functions or structures containing common elements are usually considered to transfer more easily than those lacking common elements. While the data are equivocal, it seems that there is a great deal less transfer among Piagetian operations than one would expect. The notion of horizontal *décalage,* while it gives a label to the disparities in transfer, does not adequately explain why transfer is difficult when common operations are involved. While some training data show that closely related operations transfer more easily than less closely related operations, the issue is far from settled.

6. While the acquisition of knowledge is not accounted for solely by the development of operative thinking, it is one thing to assert that Piaget gives little heed to other factors in development, it is another to say he deprecates them. Although Piaget does not ignore motivation, learning, and language, he does subordinate them to development. Although he does not say that learning is to be accounted for by development (i.e., "learning is development," as the contrary "development is learning"), he does hold that these phenomena undergo change only as a function of the controls exercised by development. It is another way of saying that

learning, language, and possibly motivation operate under the control of a genetic guidance system. The evidence would seem to support this; but once the fact is accepted, it is still not known how these systems interact. Piaget has contributed enormously to understanding these relationships, but the story is not yet told.

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HERMINE SINCLAIR

Representation and Memory

In the short sketch of the main stages of cognitive development, I mentioned only briefly an important complex of behaviors that all belong to what is called the semiotic, or symbolic, function. At the end of the sensorimotor period we observe the emergence of symbolic play, language, and, in general, activities that could not take place without some kind of representation of absent objects or events that are not taking place at the precise moment. One of Piaget's well-known examples concerns Jacqueline (Piaget 1935), who, at the age of twenty months, comes into a room with a bunch of grass in each hand: to open the door (which opens inward), she puts down the grass, turns the handle, pushes, picks up the grass again, and enters. A little later she wants to go outside; she puts the grass down again in the same way as she had when she entered the room, that is to say, at the threshold. However, she changes her mind, picks it up, and moves it farther back into the room so that it is not hit by the door when it opens. Such behavior, of which many examples have been observed, does not occur before the beginning of the second year and cannot be interpreted without supposing that some kind of representation (e.g., of the door's movement in the above example) has taken place.

Post-sensorimotor intelligence acquires a new dimension, which frees the action from the strict *hic et nunc* and—importantly, in view of what is one of the main characteristics of formal operations (the insertion of the actual into the full range of the possible)—it now becomes possible to perform one action while envisaging the performance of others. Representation enlarges the field of action immensely, both in space and in time; actual actions can be accompanied by representations which almost simultaneously encompass actions and events in the past or the present, in the immediate vicinity or a long way away.

These new representational capacities of the child in his second year are expressed in many different types of behavior. On the one hand, there is imitative behavior, not (as during the sensorimotor period) in the presence of the model, but in its absence. In fact, this imitative behavior seems to be a common aspect of all representations: in symbolic play (where the child can use objects to stand for other objects), in intelligent acting (such as Jacqueline's pushing the grass out of the way of the door's trajectory) which necessitates some kind of mental image, and in the beginnings of language. If we say that imitation is the common factor, this does not mean that representational behavior is a simple copying of reality. On the contrary—and far more than is usually supposed—the subject constructs his own representations according to his particular needs and capacities. Since the means of knowing is essentially through actions performed on reality, representation far more often reflects the way a person deals with a problem in action than a simple copy-image of the situation involved.

In his well-known studies on children's drawings, Luquet (1927) showed the existence of a period of what he called "intellectual realism," when the six- or seven-year-old draws what he knows rather than what he sees. In this period we observe drawings of faces seen in profile but with two eyes, of a field with flowers and with potatoes visible in the soil, or of trucks with four complete wheels as if they were transparent. In the same way, whatever the children cannot yet apprehend cognitively is deformed; for example, there is no coordination of different points of view, and in one drawing one can observe a table top as seen from above, with a toy car on top of it as seen from the side, and so on. Moreover, it is not only the drawings that are made spontaneously without a model that exhibit these characteristics but also those of figures that are in front of the child as he draws. Before the age of four, all closed figures (squares, rectangles, ellipses, circles, etc.) are copied as a curved, closed line, while crosses, *l*'s, curved lines, and so forth, are copied as "open" figures. But with children as young as three years of age, one can observe copies of drawings that essentially represent topological relationships (such as inside, next to, on the boundary of) which correctly represent these relations (see fig. 1).

In all education we rely heavily on representation. Actual demonstration with pupils manipulating is rather restricted in scope; and, even when applicable, its pertinent aspects are underlined verbally or through algebraic, geometric, or other notation. If the child himself represents reality in a distorted way, how does he apprehend information which the adult presents to him in a representational manner? An extensive series of experiments on memory images (observed either through drawings,

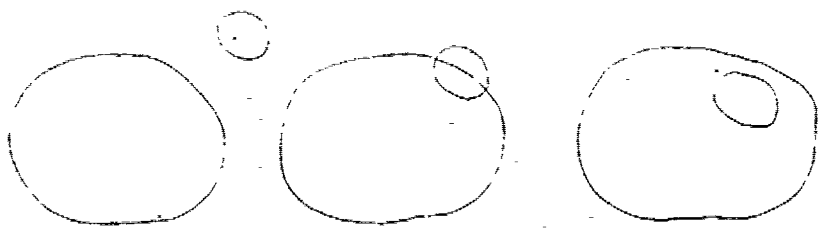


Fig. 1

gestures, or verbal explanation) has revealed many distortions—in fact, there often was as much distortion when the memory was tested immediately after presentation as there was one hour later, one week later, and even several months later (Piaget and Inhelder 1968). In every case, the deformations were due to a different way of interpreting the situation and not to simple memory factors; it is always possible to check for this eventuality by presenting similar situations that are cognitively "easier" but perceptually equivalent. For instance, in verbal memory "The table is laid by Mary and Peter" gives occasion for deformations, whereas "Peter lays the table, and Mary makes the salad" does not. Similarly, to draw a little circle on the perimeter of a bigger one does not seem to be "easier" than to draw a rectangle, however clumsily; in fact, to me the reverse would seem to be true.

Before giving a few examples of how children distort situations that they have been asked to memorize, it seems worthwhile to say a few words on the general subject of memory, another crucial factor in education.

There exist two types of memory. The first one is recognition; that is to say, of an object or situation already encountered. Recognition memory is very primitive: it exists even in nonvertebrates and, of course, in babies during the sensorimotor period. The second type of memory belongs to a higher level of development and does not seem to exist before the beginnings of representation; in fact, it is a kind of representation and consists in the evocation of situations already encountered but absent at the moment of recall. When we claim to have an excellent memory of faces but a bad memory for names, we are really saying that on seeing somebody, recognition memory works well (we recognize the face) but we cannot evoke the name, in fact, as soon as we are told the name, we recognize it just as well.

Piaget and Inhelder's book (1968) on memory concerns evocation memory and deals with a special aspect of it; that is, its relationship to different levels of cognitive development. Mental images are symbols of reality and can be used either intellectually (to solve a problem), for

play and fantasy, or for art. Mental images also serve memory for the reconstitution of past events. In this sense, memory is a type of knowledge, not attached to the present (as is perception) and not bearing directly on the solving of new problems (as is intelligence), but on the past. Developmentally, it has often been thought that fundamentally memory mechanisms are the same in the adult as in the child. There are obvious differences, because the child is not interested in certain problems and he does not understand certain situations; therefore he does not store them. However, if he does understand them, the mechanisms of retention are supposed to be the same as those of the adult, the only differences being quantitative, pertaining to span, extinction curves, and so on.

Piaget, on the other hand, maintains that there is a qualitative difference according to developmental levels; encoding and decoding processes depend on the code used by the subject, and it is precisely this code that changes with cognitive development. In fact, the amount of information transmitted by a certain number of signals depends on the number of elements and the rules of the code. To give a very simple example: if I am shown a bottle held obliquely with wine running out of it, I do not have to "remember" that the bottle was not corked and sealed. Knowing what happens to open bottles when one turns them upside down makes the information of the absence of the cork redundant. If it is true that intelligence changes the code according to which memory encodes and decodes, the same situation presented to children at different levels will carry a different information load and will be encoded in a different way.

In many experiments on memory, striking examples were found in the schematization of the situations presented. One example concerned number. The child was shown the arrangement of counters of figure 2. A collection of differently colored counters was used to demonstrate (after having asked the child to anticipate the result) that the same six counters can exactly cover all three lines in the arrangement to be memorized. Finally, the subject's cognitive level was determined by the numerical conservation test. A few minutes after presentation and once again a

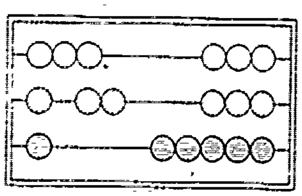


Fig. 2

week later, the child was asked to reproduce the situation by a drawing, by gestures, and by reconstituting what he had seen with a collection of counters.

Types of memory productions were closely linked to operational levels.

A first type, observed with the youngest children (four-year-olds), was the following: no numerical equivalence between the three lines and no attempt to make the lines more or less the same length. Both in the drawings and reconstitutions, the number of counters in the three lines was very different: 12, 8, and 9, or even 4, 13, and 15, and so on. But there was usually (at least in two lines) a division into groups.

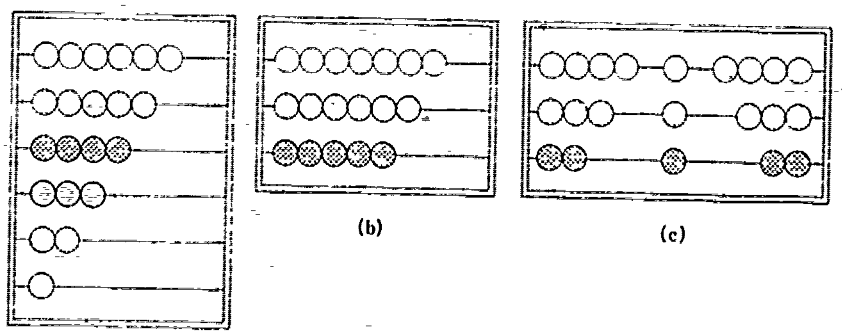
A second type was more involved (four years six months to five years): three lines with more or less the same number of counters and coincidence of their extremities (15, 13, and 13 or 7, 6, and 6, etc.). In a way, these reproductions are figuratively less "true" to the model, since in this type there is absolutely no indication of the subgroups. However, there is progress in that there is an indication of numerical equality.

A third type reveals further progress. Once again the lines have coinciding extremities, and the numerical equality is more marked: at least two of the lines have the same number. Moreover, this time the subgroups are marked, although in a peculiar way: in the lines with exact numerical equality, the subgroups are also equal, for example, twice 1 and 4 or twice 1 and 3. The synthesis between the spatial disposition in subgroups (present in an isolated manner in type 1) and numerical equality (indicated in type 2) is not yet possible.

Finally, at five to six years, numerical equality is correct; all three lines have the same number of counters. But only one child (out of eight) reproduced 6 counters per line and exactly the arrangement of the original. The other seven associated the numbers in their own personal way ($3 + 3$, $2 + 4$, and $1 + 5$; or $3 + 3$, $1 + 4 + 1$, and $1 + 5$; etc.).

All children who produced type 4 drawings or reconstitutions succeeded in the numerical conservation task (also one child of only four years and nine months).

It is interesting to see what happens to this memory some six to ten months later. Without any new presentation of the situation, almost all the children had some recollection of what they had been shown. This, in itself, is rather remarkable and indeed encouraging for educators. At the end of this long period, the two aspects, numerical equality and spatial disposition, seem to have become more separated. Again, most of the children tried to represent the numerical equality but seemed to have completely forgotten the spatial disposition. However, two new types of drawings and reconstitutions appeared that are of particular interest (see fig. 3). The first type is a *seriation:* we find lines of 6, 5, 4, 3, 2, and 1



(a)

(b)

(c)

Fig. 3

counters, placed one above the other as in figure $3a$. More often, there are only three lines as in the model, but again seriation; for instance, in figure $3b$ the top line has 7 counters, the next 6, and the last 5. The second type (fig. $3c$) is a symmetrical arrangement: 4, 1, 4; 3, 1, 3; 2, 1, 2.

As might be expected, memory of the situation becomes increasingly schematized as time goes by. However, the remarkable appearance of seriations and symmetries (which are a type of figurative classification) is more interesting than a schematization. According to Piaget's analysis of the concept of number, this concept is attained by a synthesis of the two types of grouplike structures, that of seriation and that of classification.

In another experiment, a problem of transitivity was involved. If there is more liquid in glass B than in glass A and more liquid in C than in B, is there more liquid in C than in A? However, it is not the logical problem (which is difficult to solve and to remember even after the age of seven) that I want to present here, but a curious phenomenon that reveals how even a simple action such as the pouring of liquid from one glass to another can be deformed in memory. The experiment involved four glasses—one with red liquid, one with yellow, and two empty. The experimenter pours the yellow liquid into an empty glass and the red liquid into another empty glass. Then the yellow liquid is poured into the glass that originally contained the red and vice versa, so that at the end, the contents of two differently shaped glasses are interchanged. (See fig. 4.)

When the children were asked to tell us what they had seen, the four- and five-year-olds maintained that we had poured the yellow liquid into the glass with the red liquid and vice versa. We wondered whether this was a kind of abbreviated description of what had really happened, and we showed them the four glasses with the liquids in their original positions. To our surprise, the children actually took up the two glasses that

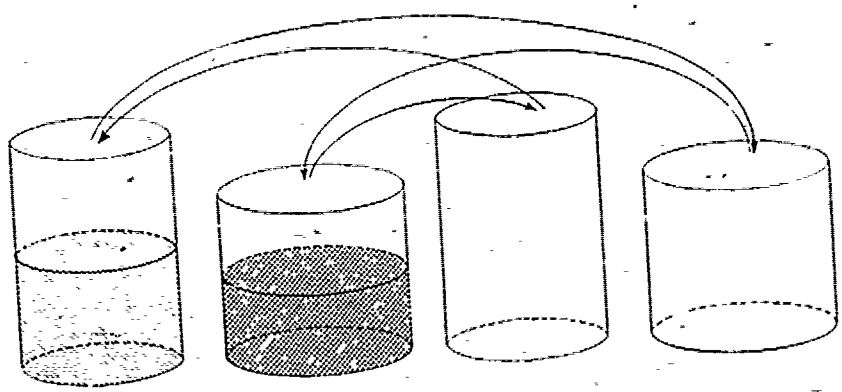


Fig. 4

were filled with liquid and tried to pour, simultaneously, the yellow liquid into the glass with the red liquid and the red into the glass with the yellow. Questioned as to whether they really thought that this could be done, they maintained their answer: "Yes, if you're clever enough." "Won't the yellow and red get all mixed up?" was our next question. Many hesitated or simply said "no." One child said, "Yes, maybe, but it will unmix itself in the end."

Another memory study concerned a double-entry table and involved wooden buttons (round and square, blue and red) glued onto a piece of heavy paper as shown in figure 5. (The experiment was designed by J. Bliss.)

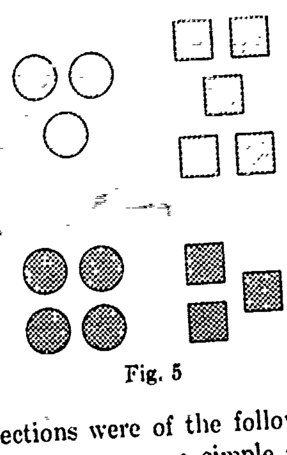


Fig. 5

The children's recollections were of the following types. A first type, found only at four to five years, was a simple agglomeration of buttons, sometimes only red (or only blue) ones. Neither the figurative disposition nor the classificatory principle was represented (see fig. 6).

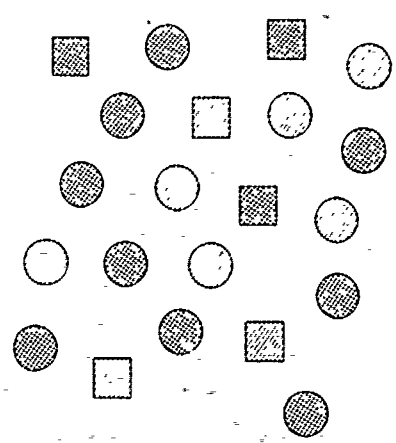


Fig. 6

A second type, more involved, showed the beginnings of a classificatory principle in the sense that two classes were present (red and blue squares for instance), but instead of an arrangement into four groups, there was a line (a circle or haphazard arrangement) made up of either just two buttons (a red square and a blue square or a red round and a blue round) or a great number of such couples (see fig. 7).

Fig. 7

A third type had the correct spatial arrangement in four groups, but again only two classes were represented (very exceptionally three) as in figure 8. Sometimes there were groups of buttons and sometimes only one.

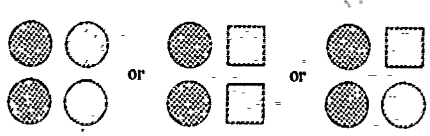


Fig. 8

A fourth type revealed considerable progress, since the multiplicative structure was present: all four classes were represented (fig. 9). How-

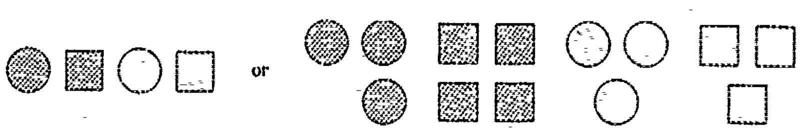


Fig. 9

ever, the double-entry-table arrangement was absent; there was an alignment either of one element of each class or of small groups of three or four elements of each class.

Finally, both the disposition and the classification were remembered, but again there may be only one element per class, which seems to represent a group of several elements (see fig. 10).

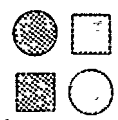


Fig. 10

It is important to realize that although the situations we presented all have a cognitive structure which can facilitate memorization, these experiments were not simple duplications of the corresponding tasks where the child himself has to classify elements, seriate sticks, and so on. In a very broad sense, *memory* also includes the cognitive structures that permit the solving of new problems. But *memory* in a more limited sense concerns only recognition, reconstitution, or, especially, evocation of events located in the past by the subject himself. In a sense, in these memory experiments the relationship between memory in the broad and strict senses was studied. In all cases, the results indicate that it is the level of cognitive development, that is to say, the particular cognitive structure of a certain stage or substage, that determines not so much the amount as the organization of the information remembered. Usually, our younger subjects do not remember "less" than the older ones; in certain cases they seem to remember more. But they remember differently. Remembering differently in this case does not mean that the younger children picked on certain details and the older children on others; it means that the total situation was differently organized according to developmental level. In this connection, it is useful to mention Piaget's distinction between a *scheme* and a *schema*. A scheme concerns the general structure of actions and operations (e.g., the scheme that permits one to arrange elements in an ordered series). As such, Piaget's schemes include such entities as "cognitive strategies," "conceptual frames," and so on. A schema, by contrast, is merely a simplified imagined representation of the result of some organizatory activity. A model unites the two: it is a schema insofar as it is a simplified representation of a particular situation, but it is a scheme insofar as it is a means of generalization.

All our findings suggest that we should not suppose that the better results obtained in our experiments by the more advanced subjects can be explained simply by the fact that they possess both more and more

accurate schemata of representation. Of course, they have encountered more situations and accumulated more of these schemata, but more importantly, they approach the situation with different schemes, that is to say, with a different cognitive organizatory capacity. The active functioning of this organizatory ability makes mnemonic encoding and decoding possible and determines its form. In this sense, the mnemonic code itself is structured and restructured along with general cognitive development.

In general, our results strengthen the educational tendency away from rote learning. It could well be that with excessive emphasis placed on rote learning, pupils would cease to use their organizatory capacity (which alone permits economical encoding and decoding) and come to rely only on a figurative, copy type of memory, which does not attain the efficiency of cognitively organized remembering. On the other hand, we do not want to equate all memory with our particular situations, which appeal specifically to this organizing mnemonic capacity. In many subjects taught in school, a certain amount of rote learning is, in the present system, inevitable. But as regards mathematics and allied disciplines, it appears that it is the concept formation itself that should be fostered by all possible means and that all representation is liable to be deformed by those pupils who are as yet incapable of a cognitive grasp of the problem involved. On the other hand, the deformations they introduce in what seemed to be a perfectly clear model can be precious indicators of their cognitive level.

Some deformations found in verbal memory are similar, although they are often less clearly linked to cognitive level. One of the difficulties in the presentation of verbal material is that a certain construction can be perfectly well understood in some instances and not at all in others. What Slobin has called *reversible* sentences are a good example of this. "John kicks Jack" is reversible in the sense that "Jack kicks John" is also a semantically possible expression; on the other hand, the permutation of subject and object in "John kicks the table" results in the impossible (or at least very improbable) expression "The table kicks John." This distinction explains many phenomena in children's comprehension of sentences such as "This is the house that Jack built" and the noncomprehension of sentences such as "This is the boy that Jack kicked." In an experiment on passive sentences, this difference was also found between sentences such as "The car is washed by the man" and "The car is followed by a truck." In an immediate-memory experiment, we found that half of the four-year-olds quite correctly repeated sentences of the second type, but without being capable of understanding the relationship between "actor" and "acted-upon." At five, these correct repetitions

without understanding began to disappear and in their place we noted different expressions that correctly indicated the relationship but did not reproduce the passive construction. Many children simply turned the passive into the corresponding active but were convinced that that was what the experimenter had said (Sinclair and Ferreiro, forthcoming).

In general, it seems that representational ability is closely linked to cognitive level, but with important differences as regards the information represented. Probably because in mathematics representation is so close to operations, in that discipline the influence is the clearest. Moreover, although as yet not much is known about the development of the different aspects of the symbolic function, it seems that this development varies more from one individual to another than does that of cognitive structures. There are important individual differences in the use of language—and the same can be said about painting, music, and acting. In contrast with mathematical notational systems, these symbolic representations can, up to a point, be dissociated from what they express. The language of poets is not more beautiful than that of other people because they have better concepts to express. Inversely, poor language and clumsy drawings do not necessarily indicate low conceptual levels.

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KENNETH LOVELL

Proportionality and Probability

It seems common for fourth formers, say, to be perfectly at home with . . . but to be utterly terrified by a problem in simple proportion.—New Scientist, 7 May 1970.

All experience in the classroom shows that the scheme of proportion is not readily available to the pupil, and since it is so important a scheme, it is worthy of some considerable attention.

Proportion

As you know, Inhelder and Piaget regard the essence of formal-operational thought as the ability to reverse the direction between reality and possibility. But we can also think of it in other ways. Since the pupil now no longer deals with the intuitable but rather with verbal elements, a new kind of thinking—propositional logic—is imposed on the logic of classes and relations. Again, formal-operational thought may be characterized as second-order operations, for the subject can now structure relations between relations as in the case of, say, metric proportion, which involves the recognition of the equivalence of two ratios. Indeed, the position was neatly expressed by Inhelder and Piaget (1958, p. 254): "In this sense proportion presupposes second degree operations, and the same may be said of propositional logic itself, since interpropositional operations are performed on statements whose intrapropositional content consists of class and relational operations." It is because the scheme of proportion, like a formal grasp of the concepts of, say, thermal capacity or energy, depends on second-order operations that the scheme is a late acquisition. Alas, the teacher of mathematics and science knows of this lateness as a result of experience, but he has not hitherto understood why this is so.

Proportion in geometric form

Piaget, Inhelder, and Szeminska (1960) argue that before a child can think about similar figures, he can directly perceive whether figures having different dimensions are similar. So the idea of proportions must, in their view, be sought in the perception of figures.

One of the many experiments that they used involved showing the child a horizontal rectangle 1·5 cm × 3·0 cm as a model and larger rectangles all of the same width, 4 cm, but varying from 6 to 15 cm in length. The model and the figures for comparison were presented in random order and the subject had to pick the large one that "looks like" the little one. It is said that in making a choice between the alternatives, intelligence is governed by perception, and the Geneva workers speak of this exercise as a perceptual estimate.

The child was also presented with the same model 1·5 cm × 3·0 cm and asked to draw a box, square, rectangle (or whatever he cares to call it), "the same but larger," on another sheet of paper. One can either avoid suggesting any length for the base of the drawing or fix it at two, three, or four multiples of the base of the model. Now in these drawings it is intelligence governing perception, and Piaget and Inhelder speak of this as an intellectual construction. They argue for the following broad stages:

1. The child is unable to make any serious effort at the tasks.
2. Efforts are usually confined to the attempt to reproduce what he looks upon as the essence of a rectangle—that is, an elongated square. Thus his drawings tend to exaggerate the length of the rectangle. When it is laid alongside a correctly proportional enlargement, he thinks the latter too high and wants to cut it down. There is no desire to measure.
3. There is now a spontaneous attempt at measurement, but the child's efforts fail because he still does not realize that it is a proportional rather than an absolute increase in size that is required, with the result that the length of the rectangle is still exaggerated in his drawings. However, if he is shown larger rectangles drawn to scale, perceptual estimates are in advance of drawings and appear to guide the latter. The subject centres alternately on width and length and appears to be trying to take into account both dimensions simultaneously and to arrive at a conscious conclusion. During the latter part of the stage, both length and height are increased, by adding an equal amount to each, in an effort to obtain the correct ratio. Once again he finds his perceptual estimates and intellectual constructions at variance, and he may alter his calculations to suit the estimates. Only in the case of simple proportions involving the ratio 1:2 are the answers correct.

This instance needs only concrete-operational thought, since one side has to be double the other.

4. The pupil begins to understand proportionality, thought influences perception, and he can draw correct constructions when height and length are in the ratio of, say, ¾ or ⅘.

Our experience at Leeds is that pupils' responses can, more or less, be placed into these categories, although the ages at which stages are reached are somewhat older than Piaget proposes. However, I do not wish to discuss more minor differences from the Geneva findings but rather to look at something of far greater importance.

Piaget's notion of scheme

You will notice that in my opening words I referred to the scheme of proportion. By the term *scheme* Piaget indicates, if I understand him correctly, the general structure of actions or operations; it is the generalisable aspect of coordinating actions that can be applied to analogous situations. So we can speak of a scheme of ordering, a scheme of classification, a scheme of proportion, and a scheme of probability. We are here dealing with general knowing, and it would be in keeping with Piaget's general position to suggest that the child would have, say, substantial classificatory or ordering skills in all areas of experience once they were available in one. This does not imply that there are no variations in the ability to use these operational schemes according to content, context, subject area, and so forth, but we would not expect strength in one and utter weakness in the other (cf. the problem of the horizontal differential). The point is that we need to know far more about the development of the scheme of proportion across a wide range of content areas. It is, therefore, my intention now to outline some further work of the Geneva school which is relevant to the scheme of proportion, in the hope that I shall encourage someone to undertake a longitudinal study involving the growth of this operational scheme across a very wide range of content areas.

Some further work

Thus I now turn to discuss some experiments taken from a more recent work of Piaget and others (1968). Although this work deals primarily with the move from contributory to well-formed functions and with their quantification, some very interesting experiments are described that illustrate the growth of the scheme of proportionality. Their findings are suggestive, but they are in great need of confirmation. However, the upshot of their findings will only be appreciated if we flavour the experiments themselves.

In one study, three "fish," A, B, and C, respectively 5, 10, and 15 cm in length, were shown to the child. He was told they were eels so that their length could increase without a corresponding increase in girth. Up to fifty "balls of meat" were available, and the child's task was to give to each fish a suitable number of balls, bearing in mind that, or on the assumption that, the strength of the appetite of a fish corresponded to its length. In a second task the fish had to be fed with "biscuits" which were represented by little rulers that appropriately varied in length. Once again the length of each biscuit given had to correspond to the length (appetite) of the fish. It will be appreciated, of course, that the meatball units were discontinuous whereas the little rulers were continuous in respect of their lengths. The following types of questions were asked:

1. If fish A gets one ball, how many balls do B and C get?
2. If fish B gets four balls, how many are needed for A and C?
3. If fish C gets nine balls, how many will A and B get?

All the questions asked using meatballs were again put to the subject, but now in terms of biscuits.

Piaget and his colleagues claim that children's responses fell into four broad stages as follows:

1. The child merely judges in terms of *more* or *less*, so that providing fish B gets more than fish A and fish C more than fish B, almost any number of meatballs or any length of biscuit will suffice. If this is termed a qualitative proportion, it is, in fact, only a kind of qualitative correspondence.
2. Numerical quantification begins in a simple form. An ordinal or qualitative property is seen between the lengths of the fish A, B, and C and the order of the quantities of food A', B', and C'. But the relations between relations, B' is to B as A' is to A and so forth, seem at this stage to be expressible only in the simplest cardinal form of $B' = A' + 1$, $C' = B' + 1$. Moreover, it appears that this stage comes a little later when using the rulers than when using the balls, which are discontinuous.
3. True metric proportions are not yet available, but the child uses preproportions that are more complex than those of the ordinal type found in stage 2. In fact, the type of preproportion used is what Suppes has termed *hyperordinal*, for one is, so to speak, halfway between an ordinal scale and metric proportionality. The intervals between A and B can be compared as *more* or *less* than the difference between B and C. If the difference between A and B is a and that between B and C is b, then the child's preproportionality is of the form a is to a' as b is to b', but in which the equality of cross products is missing.

4. Relations between relations are understood and metric proportionality is realized.

We have, of course, known for a long time that some children of ten years of age can solve arithmetic progressions. They can also work many verbal analogies of the form "toe is to foot as finger is to hand." The latter involves a preproportionality that does not demand the equality of crossed products, and they can be worked at stage 3.

I also draw your attention to five other experiments reported in the work previously cited (Piaget et al. 1968). They are very important ones, for they deal with the scheme of proportion in settings that employ physical apparatus and are germane to the work of the mathematics and science teacher. I would remind you again that Piaget and his colleagues are here studying the quantification of well-formed functions. A function is considered by them as a relation between the magnitude of two quantities, the variation in one bringing about a variation in the other in the same proportion. While their view of a function helps us in our search for the development of the scheme of proportionality, it is, of course, less general than the present-day mathematical definition of a function.

Altogether, 353 pupils were studied in the five experiments, the ages of the former varying from six to fourteen years. Unfortunately, the number of pupils taking any one experiment ranged from 41 to 116. However, the experiments involved the following:

1. The decrease in length of one side and the increase in length of an adjacent side of a rectangle that has a perimeter of constant length
2. Serial regularities of diameters and positions of rings placed on rods of different lengths which were themselves set at fixed distances apart (see fig. 1)
3. Relationship of the size of a wheel and the distance travelled by a point on its rim

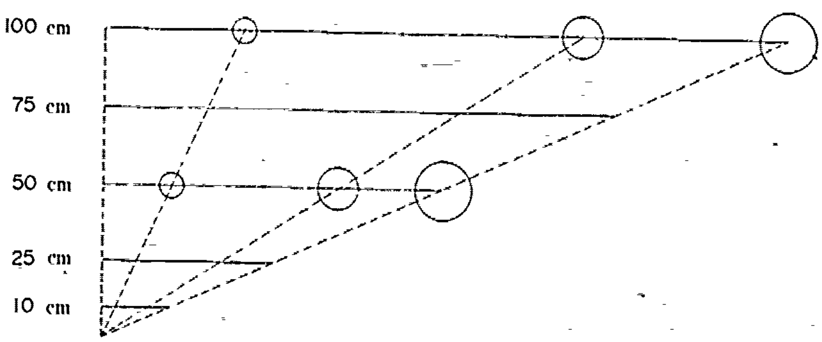

Fig. 1

4. Relationship of the size and frequency of rotation of wheels and the distances travelled by objects at the end of strings the other ends of which encircle the wheels

5. Relationship of the magnitude of a weight and its distance from the fulcrum when the arm of the balance is in equilibrium

The Genevan workers claim that a child passes through a number of rather well defined stages in the growth of his understanding of functional relationships. Once again their results are suggestive; they now need confirmation. In the first, which lasts up to seven to eight years of age, the child has great difficulty in relating the successively ordered values of one variable to those of another; that is, the successively ordered values of x and y in $x = f(y)$. For example, in experiment 1 the decrease in height of the rectangle does not necessarily bring about any increase in its width In short, there is an inability to coordinate variables.

In the second stage, which arises around eight to nine years of age, there is growing awareness of simple correspondences—that on returning to the same state one always finds the same values. For example, in experiment 1 a given height always corresponds to a given width; in 3 a certain distance is always covered by a point on the rim of a wheel of given size in rotating once; while in experiment 5 a given weight is always suspended from a point that is at the same distance from the fulcrum when the balance is in equilibrium. This, in the view of the Geneva school, is the starting point for constructing all functional variation . But at this stage two problems remain. First, there is that of comparing absolute difference. In experiment 2 this involves, at this stage, placing rings in position entirely in terms of qualitative seriations rather than in terms of relative differenc , which involves comparing the difference between the diameters of rings, say X and Y, with that between the diameters of Y and Z. Second, there remains the problem of direct or inverse compensation in questions of quantification.

In the next stage, said by the Geneva school to arise between ten and twelve years of age, although in my view the age is much later in children of average ability, the beginnings of a solution to these problems are seen. In experiment 2, the child places ring 7 nearer to ring 10 on the rods than he does to ring 1. That is, he places the rings along the rods at distances from the end that are in approximate ratio to the rings' diameters. Moreover, when the rods of length 100, 75, 50, 25, and 10 cm had to be seriated in that experiment, younger pupils had placed the rods at equal intervals, whereas ten- to twelve-year-olds realized that the distance between the 25-cm and 10-cm rods is less than the difference between the 50- and 25-cm rods. Again in experiment 5, although the distance of a weight

from the centre of the arm is regarded as one unit—what is taken as unity may change from instance to instance—all other distances in a particular instance must be calculated in terms of that unit. Finally, in respect of inverse proportion, the child in experiment 4 can connect four terms—a small turn for a large wheel and a large turn for a small wheel—in order to give equal distances travelled by the objects.

It is the search for a law of progression for the actual values of the variables which marks the passage to the fourth stage. In Piaget's view, two conditions must be fulfilled before this stage is fully reached: the pupil must be able both to handle the boundary conditions of the variables and the ratios between the successive ordered values of the variables. If the functional law is to be expressed only in qualitative form, it is sufficient merely to relate the variables: height decreases as width grows, the larger the wheel the greater the distance a point on the rim covers, the heavier the weight the nearer the centre we must hang it. The ten-year-old understands these relationships. But it is from twelve years of age onwards for Piaget—in my view twelve for the brightest and fifteen for more ordinary pupils—that the boundary conditions can be established and the intervals precisely defined.

Thus in experiment 1 the limits are between the sides of the original rectangle and a height of zero and a width equal in length to half the original perimeter; that is, as $W \rightarrow 0$, $l \rightarrow$ semiperimeter. The child has to be able to appreciate the necessary and reciprocal compensations to regulate for all the transformations as the rectangle undergoes the various changes. As far as the ratios between the successive ordered values of the variables are concerned, it is instructive to note changes in behaviour. Earlier in experiment 2 the position of each ring is determined by its rank, that is, according to qualitative seriation. It is also placed according to the relevant and corresponding fraction of the rod, so that ring 7 is placed $\frac{7}{10}$ of the way along the rod. Finally, the pupil can construct true proportionalities so that the ratio of the diameters of any two rings is equal to that of the ratio of their distances from the end of the rod. Again, in experiments 3 and 4 the final stage is reached only when the size of wheels, or size of wheels and frequency of revolution, can be put into precise proportionality with the distance travelled. Even so, according to the Geneva evidence, relations of inverse proportionality come later than those of direct proportionality.

So it seems that the child only slowly and laboriously grasps the relation between the magnitude of two quantities when the variation in one brings about the variation in the other in the same proportion. At the outset it is a mere putting into correspondence two values—for example, a larger wheel and a greater distance—or it appears in the form of some

causal dependency—for example, the weight of a piece of iron depends on its size. The final stage depends on the elaboration of formal thought, for then the ratios between successive pairs of ordered values of a variable can be handled.

Many other studies have confirmed the lateness of the growth of pupils' ability to handle metric proportion—around twelve years of age in able children and fifteen or later in ordinary ones. These studies have been carried out in a variety of content areas:

1. Inhelder and Piaget (1958), using the balance, also the rings and shadows experiments
2. Lovell (1961), using the same experiments as Inhelder and Piaget
3. Lunzer (1965), employing number series and number analogies
4. Lovell and Butterworth (1966)
   This study involved the scheme proportion using number analogies; the balance, also the rings and shadows experiments; relation between the size of the external angle of a regular polygon and the number of its sides; ratio of the areas of similar triangles given the dimensions of a pair of corresponding sides; and so forth. A principal-components analysis showed that a marked general factor was in evidence, which correlated highly with tasks involving the scheme of proportion. At the same time, however, there were variations in the level of performance of subjects across the tasks.
5. Steffe and Parr (1968), using problems classified as ratios or fractions and presented in pictorial or symbolic form

But no one has yet taken representative samples of children and traced their growth longitudinally in respect of the scheme of proportion, across many content areas. My forecast is, based on other work at Leeds, that the older the pupil the more the scheme will be available in widely separated contexts. Likewise, the abler the pupil the more likely it is that the scheme will be in evidence across different content areas. Inhelder (1969) has also pointed out, although not in relation to the scheme of proportion, that for pupils whose performance on the tests set was above average for their age, the differences between their results in one field and those in another were less than that for pupils of average and low levels of performance. Inhelder stated, "Their behaviour was more coherent in very different fields like space or time on the one hand, and logic on the other." Below-average subjects showed far greater inconsistencies. We need to know, too, what influence progress in respect of the elaboration of the scheme of proportion in geometric examples has on its implementation in nongeometric contexts. Again, we want more information on the developmental process and the specific difficulties pupils encounter. Most of the

time we have to accept cross-sectional studies, but these too often fail to attain the fleeting, transitional stages and the subtle interaction between learning in different fields. Only longitudinal studies will answer some of the questions we want to ask in respect of the scheme of proportion. Such studies alone will reveal if some contents and contexts are more facilitating, so to speak, than others and whether there are large individual differences between pupils with respect to content. For mathematics and science teachers who depend greatly on their pupils' having available the scheme of proportion, such work would be of great value.

PROBABILITY

Piaget and Inhelder (1951) have outlined their findings in respect of their studies of the child's understanding of chance and probability. I wish to mention very briefly one or two of their experiments, as I wish to draw your attention later to a recent study involving the teaching of probability.

Some of Piaget's views

One experiment involved counters, some of which had a cross on their face while the others did not. The child was shown two groups of these counters, with the corresponding number of crosses for each group. Following this the counters were turned over so that no crosses were visible; each of the two groups of counters was separately scrambled. The task for the subject was to judge which group gave the better chance of drawing a counter with a cross. The easier examples involved comparing groups containing, say, two crosses out of four counters with zero crosses out of four counters. Later tasks involved groups with, say, one cross out of two counters and two crosses out of five counters. The upshot was that children in middle childhood made some attempt to quantify probabilities, but their predictions were always made on the basis of the absolute number of counters with crosses in the groups and not on the basis of the ratio of the number of counters with crosses to the total number of counters. That is to say, the child at this stage seems to be able to compare ratios; he cannot reason in terms of the proportions of counters with crosses to total counters. ~~Indeed, in~~ Piaget's view the quantification of probability demands the onset of formal-operational thought.

But in order to tackle quantitative probability the child has to be able to handle, in addition to quantitative proportion, the permutations and combinations according to which a set of elements are grouped. For example, in one study it was made clear to the child that a bag contained twenty red and twenty blue marbles. He had to pretend to draw marbles

from the bag two at a time and predict how many pairs would be all blue, how many all red, and how many would contain one red and one blue. Pupils at the preoperational stage of thought tended to neglect the randomness of the situation and think only in terms of blues and reds. At the concrete-operational stage of thought they realized that more of mixed colour are likely to be drawn, but they could not give accurate estimates of probabilities since combinatorial operations were not available to them. But the older pupil does give accurate estimates of probability as this protocol provided by the Geneva school shows (Piaget and Inhelder 1951, p. 223):

> "More likely the mixed ones".—*Why?*—"Because you put in 40 marbles. So there are more chances of taking mixed ones: half the chances".—*Could we have all mixed ones?*—"That would be pretty strange."—*And if we draw from the bag many times?*—"Ten mixed pairs, 5 red ones, 5 blue ones".

A recent study involving the teaching of probability

An interesting study has recently been reported by Shepler (1969). He set out with two objectives: to test the possibility of teaching topics in probability and statistics to a class of sixth-grade pupils; and to construct a set of instructional materials and procedures in probability and statistics for such students.

Twenty-five pupils were chosen from a population of sixty-seven sixth-grade pupils by the school staff. All those who underwent the experimental teaching programme agreed to do so. The mean scores of these pupils on the Lorge Thorndike Intelligence Test, Level 3, Form A, and on the Iowa Tests of Basic Skills were above average: in the case of the former test the mean I.Q. was 117.7. In other words, the pupils involved were average to above average in measured ability and they had no reading difficulties. It will, of course, be realized that the ablest of these pupils must have been bordering on, or perhaps at, the earlier stage of formal-operational thought (Piaget's stage IIIA).

Shepler's report outlines earlier experiments in the teaching of probability to sixth-grade pupils and gives details of his own teaching programme. Topics include: reading and constructing a bar graph; subjective notions and vocabulary of probability; graphing data; probability of an event (one dimensional and two dimensional); performing experiments (verifying decisions made in terms of probability); and estimated probability.

A pretest was given, an excellent teaching programme lasting nineteen days instituted, and then a posttest was given. There is no doubt from the results of this study that, given first-class teaching, selected sixth-grade pupils can be introduced to notions of probability. In the posttest

almost all the questions were answered very well indeed. For example, there was a 100 percent correct response to the following problem:

> Spin the spinner (fig. 2) *two* times. What is the probability of getting a "2" on the first spin and a "4" on the second?

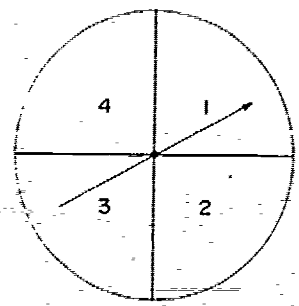

Fig. 2

Again there was a high percentage of correct responses to the following question:

> Two dice are thrown, one red and one white. The sum of the faces turning up is recorded. What is the probability of getting a sum of 2 or 3?

All questions of this type, involving small numbers, form intuitive data, and they can be solved by multiplicative classification. They can, therefore, be successfully tackled by pupils with very flexible concrete-operational thought or at the earliest stages of formal thought.

But there were just two questions in the posttest to which the correct-response rate was low. It was possible for only thirteen pupils out of the twenty-five to give the correct response to the following question involving estimated probability:

> In 6,000 spins of the spinner (fig. 3), Bob gets 2,653 reds. What is the estimated probability of getting a red on the next spin?

In this instance, although we are dealing with a finite number of spins, the

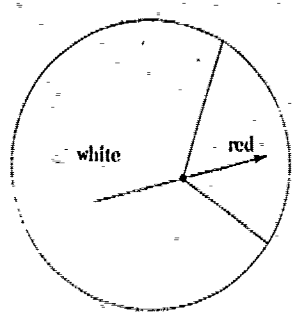

Fig. 3

number is not small and the data are no longer intuitable. Formal-operational thought is now required. Of these thirteen pupils, some may have learned to respond in rote fashion, knowing the probability was $\leqslant 1$. In the other question, which involved an estimate of a probability using large numbers, only seven pupils obtained the right answer. Piaget's developmental psychology here helps us to understand what can, and what is not likely to, be taught profitably with sixth graders depending on their general level of ability.

Probability and statistics

It is trite to say that an understanding of probability underpins students' capacity to make progress in statistics. But the point must be made. Indeed, it is worth looking at the evidence provided by Inhelder and Piaget (1958) on the growth of pupils' understanding of correlation. This work was repeated by us some ten years ago (Lovell 1961).

In the task set, the subject was presented with sets of cards on which were pictures of girls with fair hair and blue eyes, fair hair and brown eyes, brown hair and blue eyes, and brown hair and brown eyes. For example, he could be presented with a set of cards for which $a = 11$, $b = 3$, $c = 2$, $d = 8$ (see fig. 4) and then be questioned about the relationship between hair colour and eye colour on the cards.

	Blue eyes	Brown eyes
Fair hair	a	c
Brown hair	b	d

Fig. 4

The Geneva school (Inhelder and Piaget 1958, pp. 232 ff.) laid down no stages below IIIA—the earliest stage of formal thought—although in our own work we did lay down criteria for earlier stages. However, at stage IIIA the pupil can estimate probabilities as relationships between positive confirming cases and those cases that are possibly related to the characteristic in question. For example, he knows how to judge the chance that a given girl has fair hair if she has blue eyes by comparing a to b or to $a + b$. But the subject is unable to sum the positive and negative confirmings $(a + d)$ and relate these to the sum of the nonconfirming cases $(b + c)$ or to the sum of the possible instances, $a + b + c + d$.

At stage IIIB, however, Geneva claims a spontaneous relating of confirming to nonconfirming cases and to the sum of all possible cases. This adding of a and d, also b and c, marks the appearance of correlation in the strict sense of the word. The protocols given (Inhelder and Piaget 1958, p. 240) clearly illustrate the stage. In our work we found pupils needed

more prodding at this stage than the Geneva school suggests, but this might well have been due to our sample.

Once again Piaget's developmental system helps us to judge better what aspects of statistics can profitably be introduced into the elementary school, such as the graphical representation of data, simple measures of central tendency, and elementary notions of probability—the last topic with the abler sixth-grade pupils. But it will be junior high school or later, depending on the ability of the pupil, before he will be able to understand probability in a more formal sense, thus laying the foundations for statistical inference. Moreover, from the point of view of investigating the student's growing understanding in mathematics, the foundations and applications of statistical inference constitute a research area that remains wide open.

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HERMINE SINCLAIR

Number and Measurement

Two basic concepts of mathematics are number and measurement. In talking about mathematics, L. Kronecker once maintained that the natural numbers were created by God and everything else by man. By contrast, Piaget maintains that all mathematics, including number, is constructed by man! And even the construction of the simple positive numbers is a long and very complex process.

Evidently, these concepts of number and measurement should not be confused with two educationally important skills: counting and measuring. On the other hand, there *is* a close link between the concept of number—in the sense of an understanding of the series of integers—and the notion of conservation of numerical quantity, although they are not equivalent or contemporaneous. In the same way, there is a close link between measurement and conservation of length, but here again, these are neither equivalent nor completely contemporaneous. Finally, although the concepts of number and measurement may seem rather dissimilar, there is a very close relationship between them as regards their mode of construction in development.

Number

According to Piaget, the concept of number is derived from the synthesis of the system of operations pertaining to classes (with reversibility by annulment) and that concerning relations (with reversibility by

The research on which this paper is based, as well as assistance from Mrs. Susan Wedgwood, was aided by the Ford Foundation Grant no. 680-0019.

reciprocity). To handle a finite collection of objects from the point of view of their number, one has first to eliminate all qualities of the individual elements so that they become identical and thus interchangeable, whereupon it is still possible to arrange them into classes that are included one in the other (serially inclusive) such that

$$(1) < (1 + 1) < (1 + 1 + 1),$$

and so on. The individual elements must still remain distinguishable; otherwise the same element might either be counted twice or forgotten. Since all individual qualities have been eliminated, the only way to keep the elements apart is by their order (their position in space or their successive appearance in time: $1 \rightarrow 1 \rightarrow 1 \rightarrow$, etc.). Using both class inclusion and serial order, we arrive at

$$((((1) \rightarrow 1) \rightarrow 1) \rightarrow 1),$$

and so on.

Many responses of children to problem situations concerning this development exemplify this theoretical analysis of the construction of number. The difficulty of divesting individual elements of all their qualities, and thus of totally eliminating class characteristics except for their numerical value, is demonstrated in the following experiment. Starting from two collections of counters, red ones and blue ones, with the blue collection far more numerous than the red, the experimenter and the child take counters from the two collections, the child from the red and the experimenter from the blue. They take one counter at a time, always at the same moment. Having repeated this action of taking a counter four or five times, the experimenter will stop the proceedings and ask the child: "Do we both have just as many counters? We each took our counters at the same time, you a red one, and I a blue one, remember?" Very curiously, the child may answer: "You've got more counters than I have. Look, you took them from that big heap; I got them from the small heap." The child's argument seems to be based on the following thought: All counters from the red collection are red; all counters from the numerous collection are numerous (Gréco et al. 1963, p. 82).

The difficulty of understanding the serial inclusion character of number is also illustrated by an experiment designed by A. Morf (Gréco and Morf 1962, pp. 71 ff.). For the eight-year-old, it is quite clear that 9 includes 8 and that to get to 9 one has to pass through 8. To five-year-olds, however, this does not seem to be clear at all. In this experiment, there is a collection of little cubes, 7 or 9, on the table. In front of the experimenter there is only one cube, and one by one he adds cubes to his collection until he has a good deal more than 9. One first makes sure that the child knows that in the beginning he had more than the experi-

menter (9 as against 1) and also that after the adding of cubes the experimenter has more. The question now becomes whether there is a point in the proceedings when both had the same number of cubes. The five- and sometimes the six-year-olds are not at all sure. Some typical responses are: "You can't be sure; it could be first one too few and then one too many" or, quite explicitly, "There can be more and then less and never the same at all." These children seem to admit that it is possible to jump straight from *not enough* to *too many*.

Other peculiar behaviors have been found in different experiments, all showing the complexity of the construction of the number concept. In the numerical conservation experiment a preconservation stage has systematically been found (Gréco and Morf 1962) where the following phenomenon appears. There are two collections of red and blue counters, first arranged so that one is underneath the other in an optical one-to-one correspondence and then one is spread out to make a longer line. Before the age of five or six, children will affirm that there are now more blue counters (in the longer line) than red ones, or that there are no longer enough red counters to cover every blue one, or that there will be blue ones left over, and so on. If they are then asked to count the elements in the undisturbed collection, they find that there are 7—they may or may not have counted correctly, but we accept their answer. If then the experimenter covers up the second collection with a piece of paper and asks how many counters there are underneath the paper, children at this particular stage will say, "7 too." For them, there seems to be no contradiction between the two statements: "There are more blue counters" and "There are 7 red and 7 blue." Gréco (Gréco and Morf 1962) has called this phenomenon the concept of *quotity*, which precedes that of numerical quantity.

A similar behavior has been found in a quantification of inclusion learning experiment. Here we have two dolls, a boy and a girl, and the experimenter gives one of them a collection of fruit—for instance, 2 apples and 4 peaches. The child is then asked to give the other doll "just as many *fruits*, just as much to eat, so that it's fair, but give your doll more apples because he's very fond of apples." At a certain stage, children are capable of doing this and will give their dolls 4 apples and 2 peaches, keeping the total number of items constant. However, when they have assured us that "now the dolls have just as much to eat, nobody's jealous, it's quite fair," we cover the collection where they themselves have given more apples but the same number of items and ask them to count the sample collection. They say, correctly, that the first doll has six items (*fruits*) to eat. When asked if they can tell us, without uncovering the other collection, how many *fruits* the other doll has, they say,

"Eight, because I gave two more apples." This is not a simple lapse of memory, because when we ask how this is possible, since they themselves made sure that everything was quite fair, they explain: "Yes, it's fair; they have the same to eat, but one has 8—he's got 2 apples more—and the other has 6." Once again, what to us is a contradiction between two statements to them is no more than a linking of two compatible affirmations. It is only during the transition stage that children become confused and perplexed by their own answer, a factor that is particularly clear in our learning experiments. At that point it is often possible to invent a slightly easier situation where the solution suddenly becomes clear to the child. For instance, in the quantification of inclusion problem, one can then suggest that the child give only apples to his doll: in this situation he does not need to adjust the numbers of the two subclasses, and the numerical problem will become clear—2 apples and 4 peaches are 6 *fruits*, and, of course, the other doll has also 6, 6 apples.

Other curious answers have been observed by Gréco in an experiment on the acquisition of the concept of commutativity of addition. To an adult, or even to an eight-year-old it is quite clear that $7 + 3 = 3 + 7$, and it is difficult to imagine that this may be an insoluble problem to a five-year-old. The following experiment was effected by P. Gréco (Gréco and Morf 1962). Seven yellow cars (not all the same length) and three red cars are parked bumper to bumper along the sidewalk (a wooden ruler). A parking sign is then placed level with the back bumper of the last car. Now the cars are taken away and the child is asked to park them again, but this time the red ones first and the yellow ones afterwards. The three red cars are parked and one of the yellow ones when the experimenter stops the proceedings and asks the child whether the line of parked cars will exactly reach the parking sign or go further or less far. The answer may be "Oh less far this time; there's a lot of room." Even if the experimenter says, "But we're going to park all of them, the red ones and the yellow ones, just as before," they answer, "There aren't many red ones in the street; the yellow ones won't go too far . . . there is more room than before!" For such children, $7 + 3$ is not the same as $3 + 7$.

These few examples will have to suffice to illustrate the complexity of the construction of the number concept. Several other experiments show the same kind of paradoxical answers and the same confusions. The well-known numerical conservation tasks do not seem to imply an immediate and complete understanding of the nature of integers. The curious responses to several number problems seem to confirm Piaget's interpretation (resulting from his theoretical analysis) that the concept of number is derived from a synthesis of class inclusion and seriation.

MEASUREMENT

During the concrete-operational period, the child not only learns to handle, in a logically coherent way, relationships between discontinuous objects, but he also begins to be able to deal with spatial concepts. According to Piaget, this type of operational structuralization is exactly parallel to that of classes, relations, and numbers. The difference lies in the fact that in spatial operations one has to act on continuous objects into which units have to be introduced before they can be quantified. Measurement of length, for instance, implies several steps: first, a unit has to be partitioned off and then this unit has to be displaced without overlaps or empty intervals, which corresponds to a seriation; second, these continuous units form inclusions—the first bit one has measured is included in the bit that comprises two units, and so on. Thus, measurement is constructed from a synthesis of displacement and additive partitioning, parallel to that of seriation and inclusion, which constitutes the number concept.

However, this first measurement concept (length) is achieved rather later than that of number; the time lag is between six months and a year. There is an even greater time lag—two to three years—between acquisition of the corresponding conservation of length concept and the simple numerical conservations. Although the psychological construction is parallel, dealing with continuous elements is very much more difficult than dealing with discontinuous units. Moreover, there is no easy way to lead the child from one to the other, as has been amply demonstrated in a learning experiment designed by Magali Bovet (Inhelder and Sinclair 1969). Several different situations were set up with the aim of making children, who easily succeeded in the numerical conservation task, realize that they could apply the same reasoning to lengths, or roads as they were called in the experiment. Matches were glued onto tiny toy houses so that roads could be constructed (with different contours) whose lengths could be evaluated by the number of houses along them. We wondered whether children, who in a pretest had no trouble understanding that a change in the disposition of one of two lines of houses originally set up in a one-to-one correspondence did not alter the number of houses (or the numerical extension of the two collections), would immediately understand that the two roads (that is, the matches which had been glued to the houses) would also remain the same length. In this situation it is easy to ask questions alternately on the number of houses and the length of the roads. For example: "If you walk along here, do you go past as many houses as on the other road?" "Are there more

houses on this road than on the other?" "If you walk along here, do you have further to go than if you walk there? Will you be just as tired, or more, or maybe less tired?" For some children, there is no connection at all between the two types of questions: the number of houses is the same, but the roads? "The roads are different, one is much longer, because it goes further (the straight road, compared to a zigzag road); you'd be more tired, because you have further to go. . . ." Other children seemed to catch on and argued, "Same number of houses means same number of matches; same number of matches means same length of road."

However, in a second part of the experiment, children were asked to judge comparative lengths of road, which the experimenter had constructed either using equal-length or unequal-length matches; in addition, they were asked to construct roads of the same length as the experimenter's, but this time using shorter matches and following different contours or starting from a different point. In fact, using only matches of equal length means that the experimenter has already solved part of the problem for the child, who can now simply discard his intuitive solution, whereby he judges distance by points of departure and of arrival, in favor of a counting procedure where he judges by number of elements.

Having "learned" in the first series of problems that the length of a road can be judged by counting the number of matches, and having correctly solved a number of problems dealing with matches of equal length, one of our subjects was faced with the following situation: seven shorter matches making a road of length equal to one of six longer matches, the two roads being in a straight line, one directly underneath the other (see fig. 1).

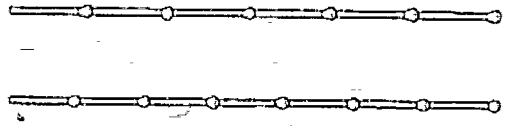

Fig. 1

This situation poses no problem even for children who do not have any conservation of length; they correctly judge the roads to be equal. After the learning procedure, however, one child announces that *A* has less far to go than *B*, since there are six matches as against seven. She explicitly refers to *A* as being less tired and to *B*'s road as being longer. When discussing the situation with the experimenter, she changes her mind several times: "Same length, because I can see it, they go just as far; not the same length, I've counted the matches, there are six here and seven there." At no point in the discussion does she refer to what would conciliate these two differ nt answers, that is, the unequal length of the matches.

The following situation (see fig. 2) was also presented: four matches in the top road in a straight line; six matches in the bottom road in a zigzag pattern; departure and arrival coinciding; and all matches of equal length. One subject answered: "The roads are exactly the same . . . except that you've put a bit more in the bottom one so that they're the same length." An involved bit of reasoning, which left the child himself rather perplexed; after a minute's hesitation, he said: "But then, why are they the same? That's what I'm wondering about—that's what's funny."

A

B

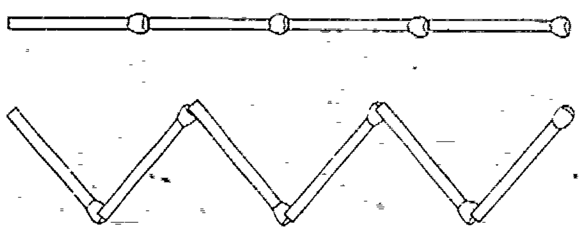

Fig. 2

The following is another example from the same learning experiment. In the situation where there are four matches in a straight line as against five in a zigzag, one of our subjects counts correctly: four at the bottom, five on top. But when we ask him about the roads, he is convinced that they are the same length: "You'd be just as tired; they go just as far; . . . ." Counting once again has no effect on his judgment of length. After the same subject has correctly solved a number of construction problems, we come back to this situation of four as against five matches. This time light has dawned and the answer is correct. But the child remembers his wrong answers and, when we ask him to explain, says, "Because I didn't count properly, because that (pointing to the extremities) came to the same place." This answer also takes some working out; in fact he *had* counted correctly—five as against four—but he had not been able to make the correct use of his counting, discarding it in favor of a judgment based on the ordinal properties of the configuration.

The situations where the child himself had to construct a road equal in length to a model, using matches of different length than those used by the experimenter (seven small matches equal five long matches) were those shown in figures 3–5. The only situation of these three which can be solved immediately by a child and whose solution gives the correct answer is the problem shown in figure 5, five long matches equal seven short ones.

The primitive way of judging length is ordinal. It is therefore not surprising that children who do not conserve length construct their road in

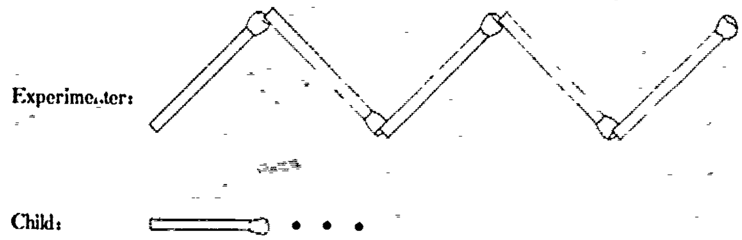

Fig. 3

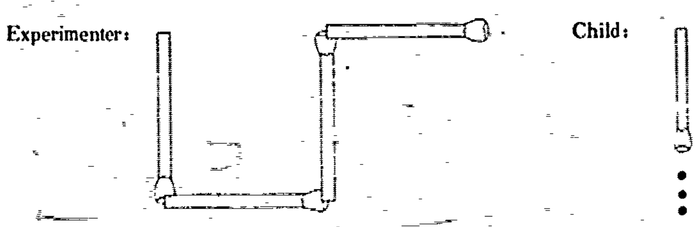

Fig. 4

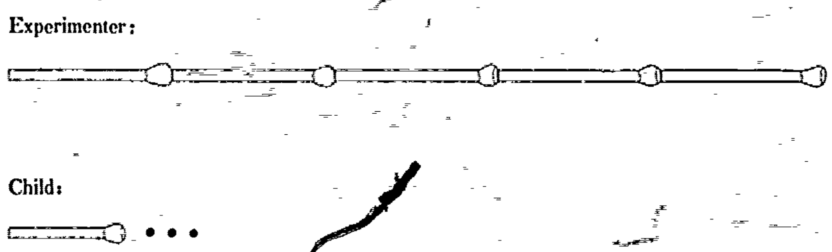

Fig. 5

figure 3 so that its extremities coincide with those of the experimenter's road—in fact they need only four of their matches to do this.

In figure 4, however, the ordinal judgment does not come into play since the roads are too far removed from each other—the road to be built is not directly underneath the model. Here we see the other primitive way of judging length, that is to say, simply by counting the elements without paying attention to the fact that the matches are not of the same lengths and cannot therefore serve as units.

Figure 5 would again be easy for a child who does not yet have length conservation, since here the right solution is immediately obvious when one uses the ordinal criterion. But after the other situations, the children in this experiment often have trouble with this problem. They count the matches in the experimenter's road, five; they take five of their own matches, put them end to end, and then decide that the problem is insoluble. "It can't be done; my matches are not right, I need matches like yours." After a while, however, they will realize that the difference in

the lengths of the matches can be compensated for by using more of them.

By virtue of having solved the third problem (fig. 5), the children are now ready to do better on the first and second problems. For the first (fig. 3), one might have expected that now the correct solution would be given immediately: seven smaller matches will make the same length as five long ones. This type of reasoning implies a grasp of transitivity, which, according to the theoretical analysis, is achieved only with the full structuralization of the system of transformations. And indeed, the children who progressed this far in their reasoning were capable of solving all the problems of conservation of lengths in our posttest (in which we used wire that was twisted and no units were involved). But many others came up with interesting compromise solutions to the first problem. Some children broke one of their matches into two pieces—thus constructing a road that did not go beyond that of the experimenter but which had the same number of "pieces"—with total disregard for the fact that not only were the pieces in their road different from the experimenter's units, but the pieces themselves were not all the same length. *Number* is certainly beginning to have something to do with length, but in a rather queer way.

The following is an example of another type of solution that seems slightly more advanced. Again wanting to equalize *numbers* in the two roads, the children used one extra match, but they put it vertically, so as not to disturb the coincidence and so that, in their opinion, "the roads go just as far, but you need more of the smaller matches than of the long ones."

A third type of compromise solution goes even further in the right direction. These children comply with the instruction that their road should be straight; they also apply the principle of "more smaller matches" and put one more match on their road, thus "going beyond" the experimenter's. However, when they look at the configuration, they may break off a piece—their road "goes too far."

The compromise solutions illustrate very clearly the difficulty of coordinating several patterns of reasoning in the problem of evaluating lengths. Judging by the points of departure and arrival is one, and in certain situations this may be sufficient. Grasping the importance of the number of units is another, which, in the case of two lengths already partitioned into the same units, is sufficient. Grasping the fact that the number of units only applies if the units are all the same and that if this is not the case then compensations have to be made is another step in the right direction. Finally, in the most difficult situation, only a coordination of these different principles coupled with an understanding of the transitivity principle will lead to the right solution.

This long analysis of the problem of length seems to provide another illustration of the complexity of the concept and to confirm Piaget's view that measurement results from a synthesis of serial displacement and additive partition, just as number results from seriation and inclusion. It also illustrates the danger of presenting children with tasks that can be solved by simple application of an, in itself, insufficient type of reasoning.

However, in the general framework of cognitive development, these findings concerning number and length give rise to some questions. As we have said, during the beginning of concrete operations the one-way mappings of the preoperational period and the functional dependencies which lack quantification and reversibility change to operations in the Piagetian sense of the term; that is, interiorized actions that are reversible, form a system with invariants, and allow new modes of composition through transitive reasoning. Although this grouplike structure and the different types of operational structuralizations that derive from it are characteristic of this whole period, certain tasks prove much easier and are therefore solved much earlier than others. In a general way, this is understandable. Concrete operations are called *concrete* because they are based on real, actually possible actions. Thus the content of a problem, quite apart from its structure, can make it easier or more difficult.

The very first conservation is that of numerical quantity—numerosity, to avoid the word *number*. The nature of the series of whole numbers itself becomes understood only gradually; in our examples we did not even touch upon what happens when number problems concern large numbers or even infinity. Another basic conservation is that of matter—a peculiar, seemingly abstract concept that is nevertheless grasped before the more precise conservations of length and weight. Now, it is understandable that conservation of the numerical quantity of a collection of discontinuous elements is achieved earlier than that of a continuous quantity. But why should the conservation of a continuous quantity (as illustrated by the problem of two balls of Plasticine, one of which is changed into a sausage, a pancake, etc.) be achieved earlier than the corresponding problem of length? The latter problem obviously does not demand a capacity to understand the abstract concept of length as a line with no width at all; the children are presented with bits of wire or very thin sticks and their width does not create any additional difficulties. In the Plasticine problem they have to deal with a three-dimensional object—why should that be easier?

There is another point. In all conservation problems one of the factors that accounts for nonconservation is the tendency to make ordinal judgments based on the ideas of *going beyond*, *overtaking*, and so on. This is,

of course, linked to the one-directional way of preoperational thought without reversibility. However, this same factor is at work in all conservation problems; why should its influence be so powerful in the case of length?

It does not seem inappropriate to finish a paper on the development of concepts of number and measurement with a catalogue of questions. In fact, it illustrates rather well what Piaget means by equilibration—the solution of one problem immediately leads to a new series of questions, which had not been envisaged before. The achievement of one stage in cognitive development implies at the same time that a new stage is in preparation. Or, as Piaget once said in answer to a question on how he felt about the future of psychology, "I am very optimistic indeed, every day I see new problems."

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KENNETH LOVELL

The Development of Some Mathematical Ideas in Elementary School Pupils

This paper deals with three issues: (1) properties of the set of natural numbers, (2) equalization of differences leading to averages, and (3) adjustments and combining of odds and evens.

Properties of the Set of Natural Numbers

A brief description will be given here of the experimental findings of P. G. Brown (1969).

Sample

Pupils were drawn from the top class of a British infant school and from each of the four classes of a junior school. The ages of the children tested thus ranged from 6+ to 7+ years in the infant school to 10+ to 11+ years at the top of the junior school. However, each school had a two-class entry, unstreamed for ability, but with classes arranged according to age, each class having an age range of approximately six months. There were roughly thirty pupils in each class. The pupils were said to form a representative sample from an urban area. Both schools used what are described as "traditional mixed" methods, this designation indicating that there was a greater degree of inquiry and self-criticism with respect to the methods employed than with the "traditional throughout" method. However, both schools made little use of structural materials.

From each of the two classes at each age level, nine boys and nine girls were randomly selected, making 180 pupils in all. At each age level, nine

standing of the following:

Identity property—addition
Commutative property—addition
Associative property—addition
Identity property—multiplication
Commutative property—multiplication
Associative property—multiplication
Distributive property—multiplication and addition

Each section of the test began with practice examples, and there then followed a number of examples to work. All instructions were given orally, with practice examples written on the blackboard where necessary.

We are not primarily concerned with the written tests. However, it is necessary to indicate the general form of the written tests, since the individually administered tests paralleled them. Examples for testing a child's knowledge of just two of the properties are given.

Identity property—addition. Put the correct numbers in the empty boxes and underline the one example that is different from the others:

$$5 + 0 = \square$$
$$3 = 0 + \square$$
$$\square + 6 = 6$$
$$8 + \square = 9$$
$$7 = \square + 7$$

Commutative property—addition. Put the correct numbers in the empty boxes and underline the one example that is different from the others:

$$4 + 2 = 2 + \square$$
$$2 + 5 = \square + 4$$
$$6 + \square = 1 + 6$$
$$\square + 3 = 3 + 3$$
$$7 + 1 = \square + 7$$

The individually administered tests

There were nineteen tests, all individually administered, that covered the areas indicated below. Sometimes there was more than one test used to examine a law.

(Marbles were only added or removed if the child hesitated.)

The following lists the possibilities for other questions:

1. Odd + 1	2. Even + 1
3. Odd − 1	4. Even − 1
5. Odd + 2	6. Odd + 3
7. Even + 2	8. Even + 3
9. Odd − 2	10. Even − 2
11. Even − 3	12. Odd − 2

STEP 6*a*. The experimenter asked:

"If you put an unknown odd number of marbles into the marble chute and then add the marbles from another odd-number chute, would the whole row of marbles be odd or even?"

If there was hesitation on the part of the child, the marbles from suitable tubes were combined in the marble chute but the sliding cover was used to prevent direct verification. In this way a test was made for:

1. Odd + Odd	2. Even + Odd
3. Even + Even	4. Odd + Even

STEP 6*b*. Here a test was made for the effect of combining the same odd or even unknown number three or more times. The procedure was the same as in 6*a*. If successful in the case of "three times," a free range of supplementary questions was asked: for example,

"If we put the same odd number of marbles into the chute five times,

Closure property—multiplication
Commutative property—multiplication
Associative property—multiplication
Distributive property—multiplication over addition

It is impossible to discuss all nineteen tests; indeed, only three will be dealt with in detail. But this will give an idea of the kinds of tasks set and of the form of analysis. It will be appreciated, of course, that the first two tests are based on Piaget's study of unprovoked correspondence and of additive composition respectively.

Commutative property—addition. The materials used were ten red and ten blue Unifix cubes, together with two sections of the Stern number track, each covered with a cardboard mask into which fourteen cubes fitted exactly.

The following method was used:

STEP 1. The child put eight red cubes, joined together, into the track, leaving a space at one end only.

Question 1. "How many blue cubes are needed to fill the space exactly? Can you find out by putting the cubes in?"

If the response was correct, the subject took all the cubes from the track and placed them in the other track.

Question 2. "Does this track hold the same number of cubes?"

STEP 2. The child put eight blue cubes at the end opposite to which the red ones had been placed, leaving a space at one end only.

Question 3. "How many red cubes are needed to fill the space exactly?"

If the correct response was given, further questions were asked.

Question 4. "Would five cubes be enough?"

Question 5. "Could you squeeze seven in?" (The child must not verify his response by putting cubes in the track.)

Question 6. "How did you work out how many were needed?"

STEP 3. Since there were three other possible positions at the end of the two tracks at which spaces could be left, the experimenter varied the

mutative relationship for addition in this situation.

STAGE II*a*. The pupil may give the correct number of cubes, but he can be dissuaded and considers that another number will also satisfy the conditions, thus indicating a transitional stage.

STAGE II*b*. This is a further transitional or semi-operational stage when pupils make an intuitive discovery without operational compositions. They are unable to express verbally the commutative principle.

STAGE III. There is an immediate and secure discovery of the correct solution. This lasting equivalence is based on the cardinal value of sets. The subject can explain the commutative principle as it pertains to the particular situation.

The results are shown in table 1, which contains the number of pupils at each stage in each class.

TABLE 1

Frequency: Class by Stage
(Commutative Property)

Class	*Stage* III	II*b*	II*a*	I	*Total*
J4	7	1			8
J3	7	1			8
J2	7	1			8
J1	3	3	2		8
Infants	2		5	1	8

Associative property—multiplication. The materials used were a number of one-inch cubes placed together to form two similar blocks A_1 and A_2, each $2'' \times 3'' \times 4''$. The layers or sections of the blocks were each of a different color. Blocks *B*, *C*, and *D* were $2'' \times 4'' \times 1''$, $3'' \times 4'' \times 1''$, and $2'' \times 3'' \times 1''$; that is, each was a layer or section of blocks A_1 and A_2.

The following method was used:

STEP 1. Block A_1 was placed on the table with its base $2'' \times 3''$.

Question 1. "How many layers are there?"

the adjustment of unknown odd or even numbers, by amounts of from one to three, and were more accurate than at level 2. But none dealt adequately with combining two unknown odd or even numbers.

STAGE III*b*. The adjustment of known and unknown numbers was very accurate compared with Stage III*a*. The combining of unknown odd and even numbers was attempted; but the replies suggest that pupils believed that the sum of two even numbers was even but that the sum of any odd number and any other number, odd or even, was odd.

STAGE IV. Relevant number substitutes were now made when combining unknown odd or even numbers. Pupils were very accurate when considering the adjustments of both known and unknown quantities by amounts from one to three marbles. When asked for the sum of two unknown odd or even quantities, they achieved the result by substituting relevant numbers for each unknown quantity. If asked if their result was true for all unknown numbers of that type, they often suggested using different numbers.

STAGE V. The frequency of odd numbers was seen as significant. Pupils answered all questions relating to the sum of three or more odd numbers and were able to generalize that the sum of any number of even quantities is itself even, whereas the sum of a number of odd numbers varies.

The results are shown in table 6, which contains the numbers of pupils at each stage in each class.

numbers are larger. Up to eight or eight-and-a-half years of age there is limited operational use of number. Thereafter, children slowly acquire greater understanding of unknown numbers, but they may be ten years of age before situations presented in an arithmetic context will be solved by generally applicable techniques. In other words, it is two to three years after number is conserved before average pupils can handle situations in an arithmetic context which call for a generally applicable method of calculation. He also points out that, as we have often found, although educationally special-school pupils sometimes achieve the same levels of understanding as their normal counterparts by mental age, in many other instances they lag far behind. However, in the practical use of money their performance is much closer to that of normal children.

When the performances of the normal children in the five individually administered tests were intercorrelated, the intercorrelation coefficients varied from 0.77 to 0.91. These are high, but their size still permits some children to be preoperative on one task but operational on another, as Brown suggests and as all other experience shows. However, the performance on the four individual tasks (the odds-and-evens task was not given) administered to educationally subnormal special-school pupils yielded coefficients varying in size from 0.52 to 0.69. This, too, confirms our findings at Leeds that the less able pupils are, the greater is the irregularity in their level of performance across tasks. Obversely, the abler pupils are, the greater the regularity in their level of performance over

After the child responded, the experimenter instructed the child to count them if necessary.

STEP 2.

Question 2. "How many layers like this (pointing to block B) would you need to make (have) the same number of cubes as in that block (pointing to block A_1)?"

STEP 3. Block A_2 was placed on the table by the experimenter with its base $3'' \times 4''$, together with block D.

Question 3. "How many layers like this (pointing to block D) would you need to have the same number of cubes as in that block (pointing to A_2)?"

STEP 4. The experimenter placed block A_1 on the table with its base $2'' \times 4''$, together with block C.

Question 4. "How many layers like this would you need to have the same number of cubes as in that block (pointing to A_1)?"

Question 5. "How did you work out these answers?"

The following criteria were used to assess the level of pupils' responses:

STAGE I. The law of associativity embodied, so to speak, in this concrete situation demands a certain capacity for spatial orientation. At this stage pupils are unable to recognize a layer or section when it is contained within a larger block.

STAGE II. There is limited use of mathematical multiplication or a restriction to counting in single units. This is a transitional stage.

STAGE III. There may or may not be some kind of physical manipulation of the blocks, but in all cases there is mathematical multiplication followed by an explanation that relates a section or layer to a corresponding part of the block.

The results are shown in table 2, which contains the numbers of pupils at each stage in each class.

TABLE 2

Frequency: Class by Stage
(Associative Property)

Class	*Stage*			*Total*
	III	*II*	*I*	
J4	4	1	3	8
J3	6	1	1	8
J2	5		3	8
J1	1		7	8
Infants	1		7	8

Distributive property. The materials for the first test include sheet 1 of paper in which there are sections *A* and *B*. Section *A* contains two arrays of crosses, 3 × 4 and 5 × 3. Section *B* contains five such arrays, namely, 8 × 4, 4 × 4, 3 × 7, 3 × 9, and 3 × 8. Sheet 2 is also divided into two sections *A* and *B*. Section *A* contains two arrays of crosses, 4 × 4 and 2 × 6. Section *B* contains five such arrays, namely, 4 × 6, 6 × 3, 2 × 9, 10 × 2, and 6 × 4.

The following method was used:

STEP 1. The experimenter showed sheet 1 to the pupil and said, "Look carefully at the crosses here" (pointing to the patterns in *B*).

Question 1. "Which of these five patterns (still pointing to the patterns in *B*) has the same number of crosses as these two patterns put together (pointing to *A*)?"

STEP 2.

Question 2. (*a*) "How did you work that out?" (*b*) "Do you have to count each cross separately?" (*c*) "Have any more patterns (in *B*) the same number of crosses?"

STEP 3. The child was shown sheet 2 with one of the two patterns in *A* covered up. The experimenter pointed to one of the patterns in *B*.

Question 3. "How many rows (or columns) of crosses would you need (in *A*) to make the same number there (*B*)?"

Question 4. "How did you work this out?"

For the second test a pegboard with two arrays of pegs, one 8 × 6 and one 6 × 3, was used.

The following method was used:

STEP 1. The pupil was shown the pegboard with the two arrays of pegs.

Question 1. "How many more pegs like this (pointing to 6 in 6 × 3) would you need to make the same number of pegs as there are in this pattern (pointing to 8 × 6)?"

Question 2. "How did you work that out?"

The following criteria were used to assess the level of pupils' responses in both tests:

STAGE I. Pupils are unable to make correct responses for various reasons, but mainly because they are unable to see a common relationship—that is, a common factor—followed by additive composition. Global assessment involving incomplete visual perception is typical of the intuitive judgments made; for example, rows with different numbers of elements are perceived as equal.

STAGE II. Although accurate use of number is made, full use of mathe-

matical multiplication is not. This limited approach usually involves counting singly, rather than applying the common factor (rows and columns with the same number of elements) or cumbersome trial-and-error methods.

STAGE III. Recognition of the common factor is necessary for full use to be made of mathematical multiplication, and stating this in terms of rows or columns is the natural consequence. The ability to apply the necessary addition or subtraction is indicative of the pupil's appreciation of distributivity.

The results are shown in table 3, which includes the numbers of pupils at each stage in each class.

TABLE 3
Frequency; Class by Stage
(Distributive Property)

Class	*Stage*			*Total*
	III	*II*	*I*	
J4	4	1	3	8
J3	3	3	2	8
J2	1	1	6	8
J1			8	8
Infants			8	8

Brown concludes, after considering all his evidence and not just the small amount reported here, that an understanding of the properties of the natural numbers develops gradually for most pupils up to eleven years of age. The paper-and-pencil tests are, so to speak, a more rigorous device than the individual work with concrete materials for testing understanding with regard to examples and closely related nonexamples. Using specifi examples (e.g., $(4 \times 3) \times 2 = 4 \times (3 \times 2)$) with concrete materials, Brown considers that understanding is reached at the following ages: closure at seven, identity at seven to eight, commutativity at eight to nine, associativity at eight to nine, and distributivity at ten to eleven years.

However, there are points to watch. In Brown's view, children's performance can be advanced or retarded up to four years compared with the norm, depending on the child; pupils can be at a preoperational stage in some tasks and operational in others; also the child achieves the operational stage with regard to all the properties tested at the earliest at about nine years of age. Moreover, an understanding of the nonexamples of the properties may be delayed for one to two years compared with understanding examples—at least for most pupils. While the relative difficulty of the items may be the same for other samples, the actual level of performance may be better or worse at any age level.

EQUALIZATION OF DIFFERENCES: COMBINING OF ODDS AND EVENS

I now turn to discuss a little of the work of G. A. Willington (1967).

Sample

Five boys and five girls were drawn from each of the year groups 6, 7, 8, 9, and 10 of a British primary school. The I.Q.'s of the pupils ranged from 95 to 106, so that one can say that as far as measured intelligence is concerned they could be described as of average ability. The parents of the pupils were mainly skilled and semiskilled artisans who were, on the whole, interested in their children's well-being, although teaching at home was unusual.

The tests

In Willington's work a large number of tests were given, some of which were paper-and-pencil tests and do not concern us. But five tasks were administered individually on Piagetian lines. Of these I would like to mention one and describe two in detail. Incidentally, the battery of tests was also administered to a sample of educationally subnormal, special school (school-educable retarded) pupils of chronological age twelve to sixteen years and mental age six years five months to eleven years seven months.

As in all experimental work of the kind in which we are interested, the responses to each test have to be placed in categories according to the type of solution offered. The information so derived is then used to establish criteria relevant to the pattern of answers. In order to check for reliability, scripts marked by one assessor should be remarked by a second.

Distributive property. I wish to say a very little about the experiment to test children's understanding of the distributive property before discussing the other two experiments in more detail. It will be interesting to compare Willington's work with that of Brown.

In Willington's study twelve boy dolls and twelve girl dolls were used. Each doll wore a garment such as a blazer, cardigan, or blouse. Each type of garment was of a distinctive design and colour and different only in the number of buttons that could be removed from the garments as required.

Tasks were set for the child which involved comparing, say, five boys each wearing three buttons with three girls each wearing three buttons and two girls each wearing two buttons. Some tasks involved inequality. But the general character of the tasks can, no doubt, be inferred from this

brief description and from the criteria for assessing the level of response.

The following criteria were used to assess the level of pupils' responses:

STAGE I. No conclusions are reached other than those based on intuitive judgments regarding the numbers of buttons—global comparisons.

STAGE II. Number is seen as relevant. The number of dolls or garments, as well as the number of buttons on each, is seen as relevant, and the totals calculated are for small numbers. But in the case of larger numbers involving, say, four dolls and ten buttons, the subject reverts to global comparisons.

STAGE IIIa. A still greater reliance is placed on number, and larger numbers can be handled. The child does not revert to global comparisons, although he may say that he did not know.

STAGE IIIb. Number is applied in a relevant way in all situations, although counting rather than multiplication may persist.

STAGE IV. There is accurate use of multiplication throughout.

STAGE V. Differences are now calculated as variations in the conditions producing equality. For example, the girl dolls are seen as a single class and equality is implicit providing that (1) the total number of girl dolls is the same as the total number of boy dolls, and (2) all the garments have the same number of buttons. Differences are seen as a result of one or the other of these conditions not being met, and any differences produced are calculated directly without reference to totals of buttons.

The results are shown in table 4, which contains the numbers of pupils at each stage in each class.

TABLE 4

Frequency: Class by Stage
(Distributive Property)

Class	Stage: V	IV	IIIa	IIIb	II	I	Total
J3	6	[illegible]					10
J2	2	8					10
J1			4	6			10
Infant 3				10			10
Infant 2					5	5	10

The pupils in the study, especially the older ones, did rather better than those in Brown's study if one dares to compare such small groups at each age level. The children were, of course, drawn from different although

comparable areas (at least on the surface); the materials were different: and different criteria were used to assess the level of response. Even so, the general trend of the results is clear in both studies, and the results are consistent with one another. These studies are useful in that they involve giving a large number of tests to a limited number of students, and one gets some idea of the stability of the level of response across tests: however, they do indicate a need for this type of study to be undertaken with large representative groups.

Equalization of differences (averages). The materials used were twenty-four wrapped sweets (these increased the interest of the game for younger pupils) and sixty identical wooden bricks (or cubes or counters). The following method was used:

STEP 1. This involved two groups of unequal size with a small numerical difference; for example, group A might be composed of six sweets and group B of four. The child was then shown the two discrete arrangements of sweets.

Question 1. "Are there as many sweets here (A) as here (B)?"

After the child was sure that the numbers of sweets in the groups were unequal, he was asked question 2.

Question 2. "Can you make them the same size?"

Whether the subject was successful or not, question 1 was repeated. After the groups had been made equal in numbers, using any method, the experimenter arranged the members of one group in a large circle and the members of the other group in a small circle.

Question 3. "If I take these sweets (large circle) and you take these sweets (small circle), who will have more sweets, you or I?"

If the subject believed that one group was numerically greater than the other, then he was encouraged to take the "larger" group.

STEP 2. This involved two groups of unequal size but with a larger numerical difference: for example, group A might be composed of six bricks and group B of eighteen bricks. The general procedure was the same as it was for Step 1, and the questioning was similar. But the two equal groups were now arranged, one in a well-spaced line and one in a tightly packed group. The question relating to conservation in spatially different but numerically equal arrangements was left out.

STEP 3. This involved three groups of unequal size: the number of members in each might be, say, 3, 12, and 6. As before, the child could use any technique for equalizing the groups.

STEP 4. Four groups of unequal size were used; the number of members

might be, say, 9, 3, 6, and 20. Any method could be used for equalizing the groups except that of aligning the members in rows.

STEP 5. Here five groups of bricks of unequal size were used, the number of members being, say, 20, 1, 11, 7, and 21. The procedure was the same as for Step 4. When the pupil had solved this problem, the experimenter removed five of the bricks and rearranged the remaining fifty-five bricks into five unequal groups.

Question 4. "Could you do it (this second problem) in any other way?"

Regardless of the method of solution adopted by the subject, the experimenter rearranged the bricks and made another set of five unequal groups comprising fifty bricks in all.

Question 5. "Could you tell me how many bricks you would put into each group if you wanted to make them all the same?"

The pupil was not permitted to manipulate the bricks physically, but he could count them if he wished.

The following criteria were used to assess the level of pupils' responses:

STAGE I. Trial-and-error forms of behavior are used to arrive at a solution to the problem. Intuitive correspondence is made by trial-and-error movement followed by counting. Or the child may make two groups each numerically equal to the smaller group and then distribute the surplus members. Numerically equal groups are sometimes mistakenly adjusted, and groups that are approximately equal, numerically, are accepted as equal. The numerical equivalence of equal groups is not conserved when the groups are rearranged in spatially contrasting forms.

STAGE II. A more analytic approach is in evidence. One-to-one correspondence can be established with lasting equivalence. Another technique used is that of accumulating all the members of the groups and then redistributing them.

STAGE IIIa. There is a progressive ability to equalize by counting or by the use of groups of arbitrary size to begin with. The general approach is to equalize the two smallest groups by counting, then to take the group next in size and equalize all three, and so on to four groups. Another approach is to adjust up to four groups so that they have the same number of members and then redistribute the surplus bricks, one to each group in turn.

STAGE IIIb. There is progressive equalization of groups by counting or by the use of groups of arbitrary size up to five groups. Pupils at this

stage are marked by their increased self-assurance and speed of working, coupled with the ability to solve all the problems presented.

STAGE IV. A nonempirical approach can be suggested by the pupils, through totalling and division.

The results are shown in table 5, which contains the numbers of pupils at each stage in each class.

TABLE 5

Frequency: Class by Stage
(Equalization of Differences)

Class	*Stage*					*Total*
	IV	*IIIb*	*IIIa*	*II*	*I*	
J3	10					10
J2	6	4				10
J1		7	3			10
Infant 3		3	7			10
Infant 2			4	3	3	10

Adjustment and combination of odd and even numbers. For the sake of clarity this task will be divided into two parts. It is an example of an involved task necessary to get at the facts. I hope you will not find the details tedious.

The materials for the first part consisted of:

1. 120 marbles
2. 18 small cardboard boxes without lids
3. An odds-and-evens board that had two rows of 10 hollows aligned in pairs along its length and a single hollow set apart in the center of the board at one end (see fig. 1)

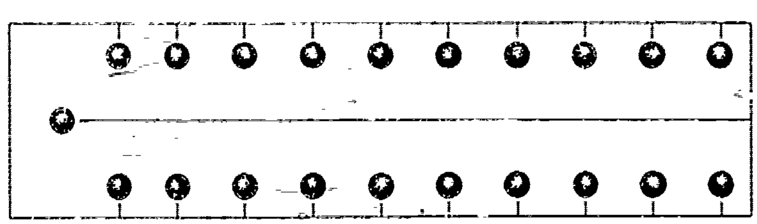

Fig. 1

The method used for the first part began with eighteen boxes laid out on the table in front of the child. They contained one to eighteen marbles respectively. By examining the boxes the child was encouraged to find out how many marbles were in the first, second, and third boxes and was

asked to predict how many marbles would be in the fifth, sixth, seventh, and eighth boxes, and then how many in the eighteenth box. When correct, he was asked how many marbles would be in boxes seventeen, sixteen, and fifteen; when these questions were answered satisfactorily, he was asked how many there would be in boxes thirteen, eleven, and nine.

The child was then shown the odds-and-evens board. The marbles from box six were divided equally by the pupil by using the board and aligning the marbles in the two rows of hollows. The experimenter asked,

"Are the two rows of marbles the same?"

When this was agreed to by the child, the experimenter said,

"The rows are even so we say that six is an even number."

The marbles from the second and fifth boxes were dealt with similarly. In the latter instance the odd marble had to be placed by the child in the single hollow at the end of the board. The experimenter added, "Five (or whatever odd number was being discussed) is an odd number because there is an odd marble left over." When odd and even numbers could be discriminated by the pupil, the first steps were introduced:

STEP 1. The child was asked about 9, 9 − 1, 9 + 1; also 6, 6 − 1, 6 + 1.
The subject was asked to locate the box holding 9 marbles and to show the number to be odd or even. After doing so the child returned the box to its proper position but left it protruding by an inch or so in order that he did not lose sight of it. The experimenter then indicated the adjacent box a place lower in the row and asked, "Are the marbles in this box odd or even?" The board was used by the pupil if he needed it, or if he wished to confirm his prediction, and the box was returned to its position in the row. This procedure was followed again after the experimenter chose the other number, adjacent to the specified number a place above in the row. After repeating the procedure for 6, 6 − 1, 6 + 1, it was continued for 8, 8 − 2, 8 + 2 and 13, 13 − 2, 13 + 2.

STEP 2. This step was intended to ascertain if a child could judge related numbers odd, or even, once a specified number was so classified. For example:

5.	5 − 2, 5 + 1
12.	12 + 2, 12 − 1
7.	7 + 2, 7 − 2
10.	10 − 1, 10 + 2

The pupil was asked to locate a specified number and decide if it was odd or even (he was encouraged to use the odds-and-evens board), and the box was then returned to the line of boxes and left protruding. After the board had been removed, the experimenter indicated another box

one or two places to the left or right of the former one and asked:

"Is the number of marbles in this box odd or even?"

This forced the pupil to make a prediction that he could not verify. After the box had been replaced in the row, a second related box was indicated for the child's consideration.

The materials for the second part consisted of:

1. Ten wooden tubes of varied length, each holding a single row of marbles. But they were accurately cut in length so that each held an exact number of marbles—half held an odd number and half an even number, with a minimum value of 5 and a maximum of 16.
2. A funnel stand with which to fill the wooden tubes with marbles.
3. A marble chute into which the contents of the wooden tubes could be transferred without revealing the number of marbles involved (see fig. 2). The marbles formed a single row along a channel of square cross section, the shoulders of which were accurately marked with an internal scale double the diameter of a marble. A removable sliding top allowed the subject to see if the extremity of a row of marbles coincided with a graduation on the scale yet prevented him from counting the number of marbles. A short length of wood that slid into the chute facilitated the reading of the scale.
4. Four opaque tumblers with lids.

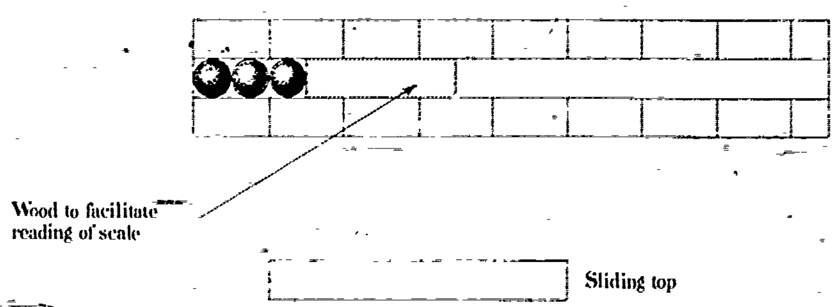

Fig. 2

The method for the second part of the task began with the experimenter asking the child to fill a tube with marbles using the funnel, to test that the tube was full, and to empty the marbles into a tumbler. The procedure was repeated using a second tumbler but the same tube.

"Which of these tumblers has more marbles?"

If the number of marbles was conserved, the subject was shown the marble chute with the cover removed and the experimenter demonstrated its use. The child was asked to place any number of marbles in the chute

either from the board or from a box, the block of wood being inserted to show more clearly whether or not the end of a row of marbles coincided with a graduation on the scale. When a judgment had been made, other odd and even numbers of marbles were similarly considered. On each occasion the pupil was asked:

"Is the end of the row opposite one of these marks?"

Providing the subject could differentiate odd and even numbers in this way, he was ready for the next step.

STEP 3*a*. This dealt with odd − 1 and odd + 1. The cover was slid into the closed position and the child assisted in filling a tube delivering odd numbers of marbles, using a funnel, and then in transferring these to the marble chute; the cover was left so that the last two marbles were visible.

"Although we do not know the number of these marbles, is it an odd or even number?"

If the child's answer was correct, the marbles were released into an opaque tumbler which was immediately covered. The subject was asked:

"Is this number odd or even?"

After one marble was taken from the tumbler, this question was repeated. If the pupil wished to return the marbles to the marble chute to verify his reply, it was done by the experimenter so that the former could not discover the precise number of marbles.

STEP 3*b*. Using tubes delivering even numbers, the child was tested for even − 1 and even + 1.

STEP 4. The procedures of Step 3 were used to test odd + 2, odd − 2, even + 2, and even − 2.

STEP 5. Wooden tubes were filled with marbles, the latter delivered to the marble chute, and the former thus classified as holding an odd or even number. Two tumblers were given unknown odd numbers, two others unknown even numbers, and they were placed to the left and right of the table respectively. The marbles in the tubes were transferred to the appropriate tumblers, always ensuring that a tumbler was empty before marbles were again placed in it. The experimenter selected a tumbler and asked:

"Is the number of marbles in this tumbler odd or even?"

If the correct answer was given another question was asked:

"If you added one more marble, what kind of number would it become—odd or even?"

PETER DODWELL

Children's Perception and Their Understanding of Geometrical Ideas

Some years ago I attempted to find out how firm the empirical base was for the statements Piaget makes about children's understanding of geometry and spatial relations (Dodwell 1963; Piaget and Inhelder 1956). The results of my enquiry were not too encouraging, and it was not pursued further. Indeed, very little attention has been paid to children's perception and their geometrical notions by psychologists caught up in the Piagetian revolution of the past decade. This neglect is the more surprising when one considers, beyond Piaget's own interest in the field, (*a*) that perception is one of the major fields of general psychological research, in which important advances have occurred in recent years, (*b*) that understanding of geometrical and other spatial concepts is seemingly intimately bound up with both perceptual and intellectual development, and (*c*) that this "interface" between perception and cognition is a major field of epistemological enquiry and has recently re-engaged the interest of certain experimental psychologists (see, for example, Neisser 1967).

Rather than reviewing research on geometrical concepts from the Piagetian point of view, I shall indicate briefly some of my findings which tend to cast doubt on the traditional Piagetian theory, then consider that theory in relation to some of the recent advances in the psychology of perception. In particular I shall be concerned with discoveries and theories to do with perceptual coding, and also with a general trend in the field which may be termed the New Nativism. What bearing might these

new ideas have on the understanding of geometrical concepts? Can the Piagetian contributions add anything to the recent major developments in the field of perception? A case can be made for saying that those developments have been very one-sided, and that the Piagetian point of view might supply some much-needed balance. Juxtaposing the two different conceptions of perception can perhaps lead to some fruitful avenues of exploration into the development of perception in children, and especially into the ways in which this development inter-relates with the growth of geometrical intuition and understanding.

Thus, rather than beat once again the Piagetian drum, I shall try to build some bridges between the Genevan school's point of view and other positions in cognitive psychology. In doing this, I shall argue that there is a valid distinction to be drawn between the apprehension, or discrimination, of forms and objects, and the understanding of their nature—or the *conception* of space and spatial relations. The weaknesses in modern perceptual theory come about largely through failure to observe this distinction. A noteworthy characteristic of the Piagetian movement is its heavy emphasis on the operational, or constructive, aspect of cognitive and perceptual functioning; hence the possibility of fruitful confluence of the two streams of thought.

Evidence for Piagetian Behavior with Respect to Geometrical Operations

Piaget's notion that the development of geometrical concepts in children follows an anti-historical order is probably familiar to most readers. The notion is that, whereas historically the earliest geometrical operations were developed to deal with practical problems of terrestrial mensuration and hence had a Euclidean character, the child only arrives at the concepts of similarity, congruence, and proportion after a long process of developing these refined concepts from more global, or general, ideas about spatial relations. Historically, the development has been from the particular, measurement-bound, practical "real world" geometry to the more general, abstract, and non-metrical relationships found in projective geometry and ultimately in topology. For the child, according to Piaget, the earliest and easiest spatial relations to grasp (in a very intuitive way) are those concerned with general features such as contiguity, neighbourhood, closed contour, and so on—that is, topological features. Subsequently ideas of perspective and "point of view"-contingent relations appear. And finally the highly specific and elaborate set of spatial operations that define Euclidean space start to emerge. As is usual in the Piagetian scheme of things, these developments are held to occur through

the agency of the child's own active exploration of, and interaction with, its environment (Piaget and Inhelder 1956).

What evidence is there that the emergence of spatial concepts is as regular, or follows as rigid a course, as this sort of theory requires? My own investigations led me to conclude that Piaget gives an oversimplified account of this aspect of cognitive development. The sorts of behavior described in his book (Piaget and Inhelder 1956) certainly can be observed in children of roughly the appropriate age, but there seems to be little coherent pattern of emergence. Thus, it is not uncommon to find children in the early school years who will give adequate "Euclidean" answers to some questions about similarity and proportion yet in other respects be still at the global, or topological, stage. I shall not attempt to document the matter here, as this has been done quite thoroughly elsewhere. The point is not so much that Piaget is necessarily wrong in his theoretical pronouncements as that the child's cognitive growth is more complex than he might lead one to believe. There is something quite satisfying—in an intuitive way—about Piaget's theories, but there is more noise in the real world than in the ivory tower. I have suggested elsewhere (e.g., Dodwell 1960, 1963) what some of the sources of perturbation might be: special interests, tuition on particular spatial relationships, and so forth, seem to be obvious candidates. Research on these aspects of the matter is completely lacking.

Operations and Spatial Relations

An example of the sorts of situation used by Piaget to study the child's understanding of spatial notions in a manner that goes beyond the mere discrimination of similarities and differences is this: the child is shown a line drawn on a sheet of paper and asked to demonstrate what will happen if the line is bisected, one of the halves again bisected, and so on without limit. A distinction is made between those who think the operation can be performed at most a very few times, those who see that it can be continued down to the physical limit dictated by the size of their pencil point and drawing skill, and those who can conceive of the operation as being in principle possible without any limit. These are identified as three stages in the understanding of "continuity", the last being the operational, correct, stage at least at the concrete level. One might of course argue whether this constitutes an adequate definition of "continuity" from the mathematician's point of view. Rather obviously it does not, but the point is not especially relevant: as a demonstration of progress in understanding the nature of lines as spatial entities whose properties extend beyond the merely perceptible, the example is illumi-

nating. In a clear sense the child who understands the possibility of subdivision without limit has a better grasp of the nature of this aspect of spatial relations than the child who does not.

Another example of a way in which the imaginative, constructive aspect of understanding geometrical ideas can be explored is the investigation of children's predictions of the shapes that are generated by cutting a solid cone in various ways (conic sections). There is a great variety in the sorts of prediction children will make, and again Piaget distinguishes several stages of understanding. The point I want to make is that extraordinarily little is known about how children develop the ability to make these predictions, the extent to which formal instruction—or informal experience—facilitates the process, what role imagery and language play, and so on. Almost certainly these topics merit closer investigation, and it may be pointed out that close adherence to the Piagetian categories of relevant responses might not be the best strategy for such work. For instance, in a more detailed analysis of children's ideas about "continuity" some attention to the (probably) related notions of compressibility and elasticity might be relevant; and similarly the study of prediction of conic sections would require ancillary investigation of ideas about solidity, invariance of shape under various transformations (translation, rotation, reflection), and so on.

I am suggesting that there is here a wealth of interesting topics for debate and empirical research which psychologists have not as yet taken up. Mathematicians interested in better methods of teaching geometry tend to ignore them too, although many fine suggestions for improving the geometry syllabus have appeared in recent years (e.g., Elliott, MacLean, and Jorden 1968). Dienes is an outstanding exponent of the imaginative introduction of advanced geometrical concepts in a simple, practical way and at an early age (Dienes and Golding 1967); but again, there is little to show that research has demonstrated the effectiveness of these methods in developing spatial comprehension. In a related field we have found almost no evidence that carefully constructed programs of instruction in arithmetic raise the level of comprehension or competence above that attained by the traditional methods of instruction (Spears and Dodwell 1970). So there is plenty of scope here for more research at both a theoretical and a more practical level.

Some Recent Findings in Perception

Potential contributions from the Piagetian point of view to the improvement of geometry teaching are so far quite meagre. I turn now to consider how recent evidence from the experimental investigation of

perception may be relevant to our general concern with the understanding of spatial concepts, and geometrical ideas in particular. There has been, in the last decade or so, a significant resurgence of interest in the topics of visual pattern recognition, the perception of objects and space, and in the nature of perceptual learning and development. Several new sources of knowledge and some newly stated theoretical positions contributed to this development, not least of which was the perfection of methods for recording the activity of individual neural cells in the intact visual system. From the theoretical side, a number of new ideas on sensory coding, and especially contour and pattern coding, added impetus to the new interest in visual space. These developments have been reviewed in detail elsewhere (Dodwell 1970); here I shall just mention some of the salient features.

The most spectacular findings from individual cell recording in the visual system come from investigations of the responses of cells in the visual areas of the mammalian brain cortex. Electrical responses can be evoked from such cells by stimulating a particular part of the sensory surface (the retina of the eye) with patterned light. Each cell responds only to stimulation of a circumscribed part of the retina, called its receptive field, and to a particular pattern of stimulation. This in itself is surprising, since the neural connections at various levels of the system, and particularly within the brain, are so intricate and complicated that one might well suppose that no simple mapping from retinal stimulation to cortical response could be found. But in fact single cortical cells do respond quite selectively to well-defined features, and the features are always straight line segments in a particular orientation. Thus some cells respond to horizontal lines, some to vertical, and others to lines in other orientations. There is a hierarchy of cell types, some responding to lines in a fixed position and orientation, some to lines in a fixed orientation but over a range of positions; some, the so-called hypercomplex units, respond best to lines of a particular length and moving in a particular direction. The main point is that we have here an elaborate and refined system for coding contour elements which is present in its main essentials at birth and must therefore be "built in" as a major feature of the visual system (Hubel and Wiesel 1962, 1963). The main evidence for such a coding system comes from studies of cats and monkeys, but there is good reason to suppose that very similar systems operate in all mammals, including man.

A rather different line of enquiry stems from the age-old problem of stimulus equivalence: how can it be that a pattern is categorized, recognized in a particular way despite the fact that on different occasions it appears in different spatial orientations and locations, and excites

different sets of receptors? Various attempts to solve this problem have been made, perhaps the most rigorous models being those stemming from the ideas of Deutsch (1955). Interestingly enough, these models show remarkable convergence with the neurophysiological findings mentioned above, and are also closely related to some of the more successful schemes for pattern recognition by machine (see, e.g., Uhr 1966).

Although there is wide agreement about the fact that in highly developed visual systems there is an elaborate, innate, primary detector system for contour elements (also undoubtedly for other sensory attributes such as color, but these are not our present concern), there has been considerable disagreement about the nature and scope of perceptual learning. On the one hand there is a mass of evidence, inspired initially by the work of K. S. Lashley and D. O. Hebb, demonstrating that experience plays a major role in the development of normal perceptual abilities in the higher mammals and man (e.g., Hebb 1949; Riesen and Aarons 1959; Held and Hein 1963). On the other hand there are theorists, particularly J. J. Gibson (1966), who argue that the nature of perceptual learning is always analytic, never synthetic, and that perception can be fully understood in terms of the global and complex analysis of sensory inputs. On this view, perceptual learning is simply a process of refining the discrimination and labelling of already existing categories. E. J. Gibson (1969) has recently extended these ideas and interpreted a great deal of the existing experimental literature in terms of them.

The New Nativism

It is the Gibsonian view, reinforced by the increasing knowledge about sensory analyzers at the neurophysiological level, which I would term the New Nativism. The neurophysiological findings do not force one to a Nativistic position, since the sorts of coding and analysis so far discovered have been basically simple, and far removed from what we understand as object and space perception. The detection of pattern elements does not itself constitute a system for the recognition of whole patterns, or *Gestalten*, and there is still scope here for the operation of a mechanism of synthetic, or constructive, perceptual learning. Yet, according to the New Nativism, every sort of perceptual learning is concerned only with finer discrimination, more exact detection of the " tinctions between different stimulus patterns: as J. J. Gibson puts it, the role of perceptual learning is to bring perception more and more into *correspondence* with stimulation, not to build new perceptual and cognitive categories.

This is a highly "stimulus bound" view of perception, and seems not to accord with what we know about the perceptual foundations of cognition.

For example, learning to read does not consist in learning to discriminate between all the different typefaces that might be used to print a book, let alone all the varieties of handwriting that are possible! Yet it is difficult to see what other position the Gibsonian view, strictly interpreted, could lead to. So we can say that one problem for the New Nativism is that it tends to disrupt that intimate connection between perception and cognitive-intellectual activities which has been a major cornerstone of empiricist theories of factual knowledge among both philosophers and psychologists. Clearly, on such a view it would be difficult to discover a fruitful link between perception and formal or abstract geometry.

One important aspect of the New Nativism is the extensive research on pattern perception in neonates and very young infants which it has engendered, much of it summarised by Gibson (1969). According to the common interpretation of such findings, evidence that a very young child can discriminate between patterns is evidence for an innate processing mechanism. Apart from some severe reservations about the quality of such evidence—difficulty in replication, contradictory findings, use of infants several months old (in which case there would have been extensive opportunities for perceptual learning)—one may point out that it proves far less than the proponents of Nativism may claim. Take, for example, experiments on "looming". In such experiments the observer is faced, typically, with a screen on which a shadow is cast. By one means or another the shadow is made to grow rapidly in size and evokes a "startle" response from the observer. Obviously the situation is analogous to one in which a solid object rapidly approaches the observer, in which case startle and/or avoidance would be appropriate and adaptive. Very young infants have been shown to make such responses to looming shadows (e.g., Bower 1969), and this is correctly interpreted as demonstrating their ability to respond to a complex optical array and its changes over time. However, the temptation is strong to attribute to this situation more than is warranted. A more-or-less reflexive response to an optical array tells us what the organism is capable of responding to, not what it understands. The point is made clearly by pointing out that the young of ground-nesting birds will make an appropriate "startle and freeze" response to a crude cardboard model of a hawk (short neck, long tail) passed over their heads. If the direction of movement is reversed, so that the model is more like a goose (long neck, short tail), no startle response is evoked. The interpretation is that in one direction of motion the model shares certain critical ("sign stimulus" according to Tinbergen 1951) features with a moving hawk, and the response to these features is innate. It can well be argued that responses of infants to looming shadows are evidence that they too demonstrate the sorts of "elicited" response studied

by ethologists in lower forms of life, rather than that they have an innate grasp of object perception and spatial relations.

Geometry and Perception

Just as learning to read is more than merely learning to discriminate between different letters, so too learning to perceive is more than learning to discern particular features in the visual world. Learning to read involves learning how words, sentences, and paragraphs convey information at a high level of abstraction. Just so, perception can involve forms of cognition which transcend the simple analysis of "stimulus information". Were this not so, geometry as a mathematical discipline would be entirely divorced from the geometry of perceived objects. It is true that, at a certain level, the treatment of geometrical operations bears little obvious relationship to the spatial ideas and intuitions on which those operations were originally based. At the same time, however, I think that most mathematicians and cognitive psychologists would agree that the perceptual substrate of geometry is real enough. At least so far as children's understanding of geometry is concerned, the first steps certainly are taken within the context of concrete, perceptible objects and drawings. It is very much to Piaget's credit that he has attempted to explore in a variety of ways this borderland between perception and cognition, a task that rather few other psychologists have essayed.

To show clearly how closely perceptual questions are tied in with the development of geometrical ideas and operations, we may consider the question of symmetry. In one sense symmetry can be a simple perceptual phenomenon; yet can it be *purely* perceptual? Perhaps there is no sensible answer to that question, but at least we can say that the detection of symmetry in a visual pattern can be accomplished without any elaborate linguistic or conceptual tools. For example, children might be taught the idea by ostensive definition, sorting patterns into different categories according to symmetry, and so on. But how are questions to do with the detection of symmetry related to the *concept* of symmetry? Obviously detection does not exhaust the topic: we might, for instance, want to know under which types of transformation symmetry is preserved; there are abstract instances of symmetry (in logical or arithmetical relations, for example) which have no perceptual referent. So, the initial apprehension of the notion of symmetry may be through perceptual instances, through learning to detect particular features in visual patterns, but few of us would argue that this is *all* it is. How does the general notion develop? What linguistic and manipulative skills are necessary and suffi-

cient conditions for its acquisition? Which ones are even relevant? We do not know.

We might guess that the simplest forms of perceived symmetry (let us say, about horizontal or vertical axes) are inbuilt and a function of some coding operations; in this connection it is interesting to note that mirror-image confusions are common among children and also can be observed in discrimination learning in animals at many phyletic levels. To grasp the full possibilities, however, requires more than coding or analytic discrimination learning. What forms of synthesis are needed? What is the relevant border between perception and operation, or cognition?

From what was said earlier about the New Nativism, it should be clear that this type of perceptual theory is unlikely to supply answers to the problem. Simply pointing to that fact might do something to redress the suggested imbalance in recent perceptual theory, but helps us not at all to answer the questions posed at the end of the previous paragraph. Piaget's own more strictly perceptual work (Piaget 1968) is perhaps closer to the mark, but again fails to show convincingly how the discrimination of pattern and spatial attributes is connected with "cognitive" space or with geometrical concepts. The treatment of geometrical ideas (Piaget and Inhelder 1956) is really an exploration of this "cognitive" space, and gives fascinating glimpses of the sorts of difficulty children have in elaborating it; but again it does not adequately relate the perceptual basis to the space which develops from it. Nor does it deal specifically with the constraints that perceptual coding might place on this development. The special roles of language and symbolism, the antecedent conditions necessary for the development of spatial understanding and imagination, and so forth, are similarly not dealt with. It is a remarkable fact that, whereas a good deal of work has been done on the nature of conservation and the training conditions that affect it (see Beilin's review above, p. 85 ff.), nothing similar has been done for the equally acute and interesting topics of geometry and spatial relations.

Conclusions

There is no extensive literature in the Piagetian tradition on geometry and spatial concepts to review, no hotly debated issues at either the theoretical or the experimental level on which to make judgements. To a remarkable degree this field has been neglected in the flood of experimental analyses on cognitive development of recent years. So my endeavour has been to show how unjustified such neglect is, to see where

research might yield a valuable harvest rapidly, and to point out how relevant such work would be, both in throwing further light on the intellectual functions that concern us here and in redressing a certain imbalance currently to be observed in the field of perceptual theory.

These polemics may not be of immediate use to the mathematics educator, but one hopes they may provide the stimulus to further thought and research on the important question of geometrical imagination and understanding.

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KENNETH LOVELL

Some Studies Involving Spatial Ideas

This is the only talk I am giving devoted to work in the spatial field. I shall attempt three things. First, I want to take just two of a number of tasks recently given to pupils of elementary school age and spell out the results in some detail to illustrate the nature of the responses made by pupils and the stages in thinking found. In none of my other talks do I give much in the way of examples of protocols. Because this is not an experiment reported by Piaget and his colleagues, the detail is necessary to help you set up your own interview techniques and analyse the results you obtain. Second, I want to mention a task referred to in another of my talks, which was used by Lunzer, in order to bring out differences between concrete- and formal-operational thought in the spatial field and to throw light on an issue well-recognised by elementary school teachers, that pupils confuse perimeter and area. Third, I want to illustrate, by taking an example from the spatial field, that mathematical ideas are dependent on the growth of schemes, which themselves evolve because of their own inherent functioning through the spontaneous experiences and actions of the child.

Area

The task involved the use and manipulation of measuring instruments, leading to the comparison of the areas of two shapes that differed markedly in appearance.[1]

Materials. These consisted of a blue square of side 8 inches, and an orange rectangle 16 inches by 4 inches. The measuring instruments were

1. The study was carried out by Mrs. M. Caltieri.

eight small, green, right-angled triangles, each being one half of a rectangle, 4 inches by 2 inches, cut diagonally. (See fig. 1.) There were obviously insufficient of these to cover, entirely, one of the larger figures.

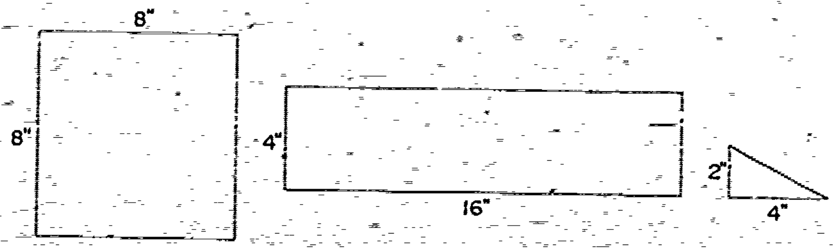

Fig. 1

Subjects. Teachers selected fifteen pupils from each of the age groups five, six, seven, eight, nine, and ten (or ninety subjects in all); the selection was made so as to give a representative cross section in terms of ability at each age level. The school drew children from a very mixed socio-economic area. About one-quarter of the pupils came from professional or semiprofessional homes, with the remainder coming from homes in which the fathers ranged from highly skilled to unskilled workmen.

Method. The main questions posed are now indicated, although it must be made clear that supplementary questions were asked in order to clarify some point or to elucidate, further, the child's thinking.

> "Here is another game for us to play. Look at this orange shape and this blue one. Which of the two has more space?"
> "If both were fields, which would have more grass?"
> "Will these small green tiles help you to find out?"

If the child gets so far with the tiling and then points out that there are not enough tiles, say:

> "How many tiles have you put on there?"
> "Can you tile the rest of it?"

The questioning then takes rather different forms according to the answer given. If the child is unable to make any suggestion, say:

> "How many tiles do you think you need to cover the whole of the piece?"
> "Why do you say that?"

If the subject says that half is covered and that he will need eight more small green tiles, say:

> "How do you know you have covered half?"
> "Is there any way in which you can prove it?"

If, however, the child replies to the question "Can you tile the rest of it?" by saying that he can do so, then say:

"How can you do that when you have no more tiles?"

If he simply pushes the whole of the tiles across, en masse, without attempting to mark the boundary of the part that was originally tiled, say:

"How do you know that these are covering the space that was empty before?"

There must be some definite strategy employed by the subject to show that he is fully aware that the tiles will cover only one-half of either larger figure and that in order to express its area in terms of the unit triangle there must be some form of iteration. If he argues that he has covered one-half of the large figure, he must show conclusively that it is one-half. If one of the large figures is successfully tiled, then say: "What about the other shape?" There should then be a similar tiling procedure. Finally, the child is questioned as to the number of green tiles needed to cover the whole of each of the large figures and which has more space.

Stages and protocols

STAGE I. There is little or no understanding, difficulty in pursuing the inquiry, and no attempt whatever at tiling. Only two pupils were at this stage, one five- and one seven-year-old.

STAGE IIA. Judgments are based on simple intuition and perception. Even if tiling is attempted, pupils are either unable to manipulate the tiles successfully or they tile without understanding what they are doing. Those who do tile are unperturbed by the fact that there are insufficient tiles to cover the whole, as would be expected, since they attach no significance to the act of tiling. They usually say casually that there are not enough tiles, and when asked again which has more space they repeat their original intuitive judgment.

S.H. (5.2): "Because it's big and fat" (pointing to the square).
B.C. (5.4): "It's more squarey".
J.C. (5.4): "Because it's longer".
K.I. (5.0): "Because it's big like a door" (pointing to the rectangle).
C.B. (6.6): "Because it's taller".

None of the subjects at this stage volunteered to use the small green tiles. When prompted to do so they obligingly tried, but most spread them out haphazardly over the large shape.

K.I. (5.0): "They all go on different shapes".
A.N. (5.11): "I can't put them on, they don't fit together".

J.F. (7.2): "These are hard ones to put on, they all have different edges".

Those children who attempted to tile did so until they had used all eight tiles and then said, typically:

J.C. (5.4): "I haven't enough tiles, I can't do it".
K.C. (6.11): "No good, no more".
M.McG. (7.8): "There are only eight tiles. You can't do anything else".

When asked, "Do you know how many tiles I'd need to cover all of it", M.McG. replied, "No. There are only eight. You can't find out, can you, when there are only eight. Anyway that orange is too big for them".

Stage IIB. There are now transitional or intermediate responses, trial and error methods, but lack of generalisation. When asked to say which of the two large shapes had more space they were loath to give a wholly intuitive answer. They seized upon the tiles and began to use them, confident that in some way they would provide an answer. When they realised eight tiles were insufficient, they were baffled. The conflict that ensued resulted in most of the pupils' abandoning the effort to prove the equality of the number of tiles required by trial and error, and stating at this point that the two large figures had the same space but they did not know why.

G.K. (6.1): "Can I use these little green tiles to find out?"
N.G. (6.4): "I think there're both the same, but I'm going to find out with these little green ones".
J.C. (7.11): "They're both the same, at least I think so. I know how to find out, I can use these green ones".
C.L. (8.0): "I will see how many of these green ones will go on both of them, can't I?"
R.D. (6.0): "Now I don't know what to do. I've only got eight. I thought if I covered them both with green tiles and they both had the same number on, then they'd both have the same space".
C.B. (8.2): "There aren't enough tiles. I don't know what to do".

The experimenter then asked C.B. if he could say how many more tiles would be needed to cover all of the blue space. C.B. replied:

"No, because there aren't enough".

Stage IIIA. When the tiles were all used, pupils at this stage declared that they would need another eight to finish tiling the square or rectangle. They were all sure of this and equally sure of the equality of

the spaces in the two shapes, but they were unable to prove that the part to be covered was in fact one-half of the whole shape.

C.C. (6.11): "They both have the same".

Experimenter: "Did those green tiles help you?"

C.C.: "Yes, 'cos I put eight here and I wanted another eight, and I put eight here and I wanted another eight".

Experimenter: "How do you know you wanted another eight?"

C.C.: "Because it's the same as the other half".

Experimenter: "Is there any way you can show me it's the same as the other half and that eight will fit on?"

C.C.: "Yes, you can see it's just the same size as the other half".

STAGE IIIB. Children were now able to show, in one way or another, that sixteen tiles would be needed. Some moved the triangles singly, others in pairs, while others moved all eight together, first marking the mid-way line in some way. In a few instances the method used was unique.

Results

The results are shown in table 1.

TABLE 1

Frequencies: Stage by Age

Stage	*Age* 5	6	7	8	9	10
I	1	0	1	0	0	0
IIA	11	7	4	3	0	0
IIB	0	7	8	3	1	0
IIIA	3	1	2	2	3	0
IIIB	0	0	0	7	11	15

A further task

Another of the many tasks given to these subjects is now briefly mentioned. The results suggest some omissions in the teaching.

In this task the eight-, nine-, and ten-year-olds were interviewed individually as before but after each had carried out the task just described. Each pupil was presented with the 8-inch blue square, the orange rectangle measuring 16 inches by 4 inches, a foot ruler, and, later, a small piece of card measuring 4 inches by ¼ inch. The teachers concerned claimed that all these pupils "had met the procedure of finding the area of a square or rectangle using a ruler and multiplying the number of units of length by the number of the same kind of units of breadth".

There was at first a discussion with each subject to ensure that he knew

the meaning of the word *area*. The experimenter then said, "If I give you the orange piece, the blue piece, and this ruler, can you tell me what is the area of this piece?" Nothing further was said to the subject, it being expected that those who could do so would measure the lengths of the sides and express the area of each figure as 64 square inches.

To each child, whether or not he could accomplish the task using the ruler, the experimenter then put these questions:

"Must area always be measured in square inches?"

"Would you measure the area of a football field in square inches?"

"What would be a better unit for measuring the area of a field?"

"If your ruler had only centimetres marked on it, what units would you use to give the area of these shapes?"

"Now suppose you had just this little piece of card (4 inches by 1/4 inch). Could you find the area of the square and the area of the rectangle just using this as a measure?"

The ruler was withdrawn. If the child said in effect that he did not know how long the piece of card was, the experimenter replied:

"Does that matter? Can you use it in some way to find the area of these two shapes?"

The results suggest that the pupils did less well in calculating the areas of the figures than in comparing their areas, which argues in turn for another look at the teaching these pupils had experienced. In the case of the eight-year-olds, seven of the fifteen could find the areas of the shapes using a ruler. Of these, four were at stage IIIB in comparing areas, one was at a very good stage IIIA and almost at stage IIIB, while the other two reached stage IIA in the first task described—but both were aged eight years and eleven months and were the oldest pupils in the age group. When it came to measuring areas with the small card, only two of the seven were successful, and both were at stage IIIB in the comparison of areas and in other tasks given. Both pupils were regarded by their teachers as able children. Three pupils at stage IIIB in comparing areas were unable to make any attempt at calculating areas.

In the next age group, nine of the fifteen were able to express the areas of the shapes in square inches using the ruler. All of these were at stage IIIB in comparing areas and at a similar stage in other tasks. Seven of these subjects, and no other child at this age level were able to express the areas in terms of the unconventional measuring instrument. Their only difficulty was in knowing what to call the units. Again in the ten-year-olds, eight of the fifteen pupils, all at stage IIIB in the comparison of areas, calculated the areas of the shapes in square inches using the ruler. Moreover, all eight, and no one else at this age level, calculated the areas

in terms of the unconventional units. The other seven of the ten-year-olds, all at stage IIIB in comparing areas, were unable to calculate the areas of the shapes using either ruler or small card.

When we examine the number of pupils at stage IIIB in the task involving the comparison of areas and the number able to use a conventional ruler and an unconventional card to measure the area of a figure, it does appear that there has been too much emphasis on a rote procedure involving measuring the lengths of the sides of a square or rectangle (in the same units) and finding the product, and insufficient practice in unit iteration. It will be appreciated that teachers can help their pupils to understand that the area of a square or rectangle is given by the product of the lengths of its sides when it is based on the successive application of a unit area within a larger area. Indeed, this is the way to proceed with children of the age we have been considering, for it requires formal-operational thought to make a direct transition from length to area by way of an arithmetic calculation. There were pupils at stage IIIB in comparing areas who seemed to have had no experience of unit iteration and the calculation of area per se.

The calculation of an area by a direct transition from length to area using an arithmetic calculation amounts to reducing the area to two infinite sets of lines with each member of one set being perpendicular to each member of the other set. That is, we have a grid with lines infinitesimally close. While the child can be taught that the area of a square or rectangle is equal to the product of the lengths of its sides, it is only intelligible if the area is reduced to infinite sets of perpendicular lines. There is, prima facie, evidence for suggesting that a number of pupils at stage IIIB in the task involving the comparison of areas were unable to calculate the areas of the shapes with rulers and with unconventional units as too little experience had been given in the classroom of unit iteration.

Conservation of Perimeter and Nonconservation of Area and Vice Versa

Lunzer (1968) has described two experiments that neatly bring out the difference between concrete and formal-operational thought in the spatial field. In essence, subjects were presented with the following:

1. A wooden board fitted with nails arranged in suitable array so that they could form the corners of a square or rectangle and a square of side 25 cm being marked out with a heavy black line.

   The pupil was shown this square with a closed length of string around it; the string was then moved to form, in turn, a series of

rectangles. Square *a* became transformed into rectangle *b*, with the perimeter remaining constant but the area enclosed by the string changed. (See fig. 2.)

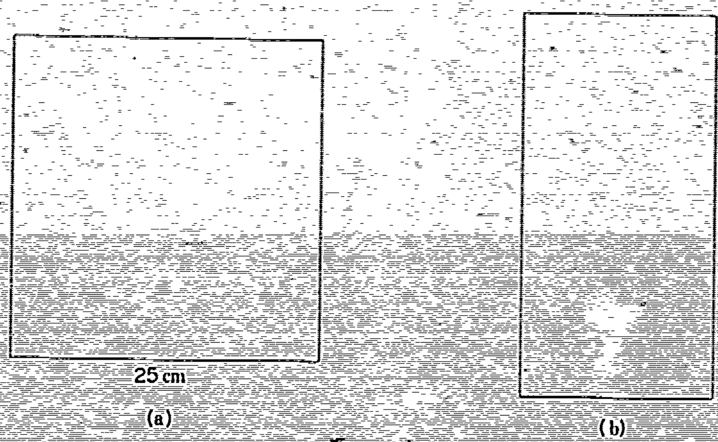

Fig. 2

2. A square of green card of side 12 cm (fig. 3*a*)
   Five further figures were cut so that the triangle taken from the bottom right corner was transferred to the top right-hand corner (fig. 3*b*). The triangular corner in the five new shapes increased in height from 2 cm to 10 cm. In this task, area is conserved but not perimeter.

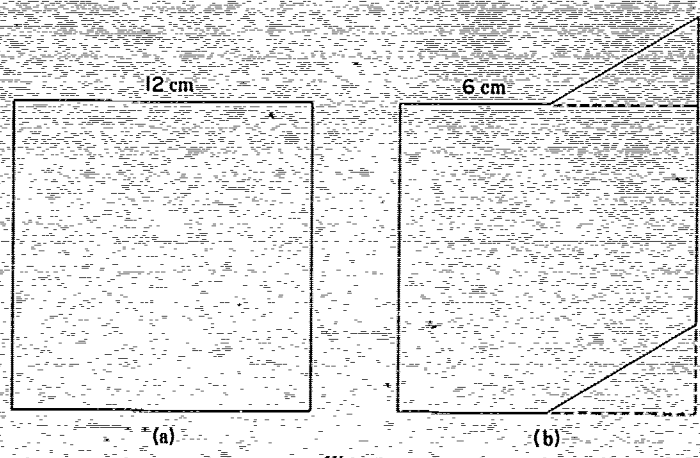

Fig. 3

Three hypotheses were proposed by Lunzer. First, subjects at the stage of concrete-operational thought would conserve both perimeter and

area in both tasks. Since an increase in height in task 1 is accompanied by a decrease in width and since the triangle changed places, so to speak, in task 2, children at this stage of thought would regard conservation of area and perimeter as logically necessary in both tasks. Second, children at the stage of concrete-operational thought would ignore the marked perceptual effects, especially in task 1, where it became obvious to the adolescent that the rectangle became thinner and thinner and that ultimately its area tended to zero as the height tended to the semiperimeter of the square. Third, at the stage of formal-operational thought subjects could generalize. That is to say, they accepted that if area (or perimeter) changed when there was a marked transformation, the same attribute must also have changed when the transformation was small, even if the other attribute did not.

Lunzer presented these tasks, individually, to two groups of subjects. One group of eighty children in Geneva ranged in age from five to fifteen years. The other group in Manchester consisted of sixty children equally divided into age groups of nine years, six months to ten years, six months; ten years, six months to eleven years, six months; and fourteen years to fifteen years. In both cities and in both tasks, the results generally confirmed the hypotheses. In task 1 there was, among children at the stage of concrete-operational thought (age eight to eleven, approximately), a clear tendency to conserve both perimeter and area in spite of the perceptual evidence to the contrary in the case of area. Lunzer claims that even when such subjects measure the dimensions of a particular figure and find the area to be smaller, they stoutly maintain conservation of area for other figures. But at the level of formal-operational thought (from thirteen years of age upwards) subjects recognised the decrease in area of the figure when the transformation was considerable and generalised this decrease, spontaneously, to all such transformations, although perimeter is conserved.

In the case of task 2, area and perimeter were conserved at the stage of concrete-operational thought, although among older pupils at the most advanced levels of concrete thought there was the beginning of dissociation between area and perimeter. At the level of formal-operational thought, the subject completely dissociated area and perimeter and realised that the former is conserved but the latter is not.

At the stage of formal-operational thought it will be recalled that subjects can consider a number of hypotheses simultaneously. Whereas for the child at the stage of concrete-operational thought there is either conservation or nonconservation, at the stage of formal-operational thought he can consider the hypotheses of conservation and nonconservation and confirm them. Again, Lunzer points out that in task 1 most subjects at

the concrete stage of thought realised that the area of the rectangle was zero in the limiting case. But he argues both from his evidence and from that of Bang (Bang and Lunzer 1965) that children at this stage look on the limiting case as quite different. The string is present but the area has gone, and conservation no longer applies. But at the stage of formal-operational thought, area and perimeter are examined as objects of thought in their own right.

Lunzer also draws attention to another development in geometric situations at the stage of formal-operational thought—namely, an ability to go outside a given figure. In the task shown in figure 4, subjects were provided with cards that exactly covered the area ax. If these cards were laid across the top of the figure so as to cover the area $x(a-x)$, they would obviously protrude beyond that area and also cover the area x^2. Those at the stage of formal-operational thought reasoned that the area of the shape had decreased by the amount of the small area x^2. But the pupils at the level of concrete-operational thought did not know how the card, after having been placed over the area ax, could help in measuring the area of the rectangle. But when some of these pupils—and more particularly those at a stage intermediate between concrete- and formal-operational thought—were given a card that exactly covered the area $x(a-x)$, they did understand that it was smaller than the area of the rectangle ax. In short, pupils at the stage of formal-operational thought can handle the situation by going outside the given limits of the figure; those not at this stage have to stay within.

Piaget, Inhelder, and Szeminska (1960, p. 192) give a clear example of this ability to go outside the bounds of the provided figure. They asked subjects to copy a given triangle using a ruler or other measuring

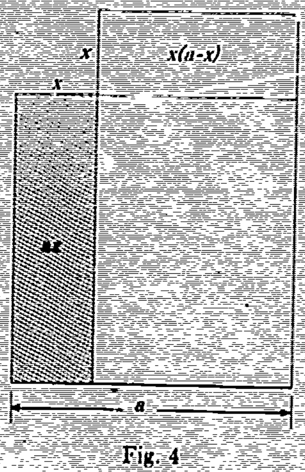

Fig. 4

instrument. At the upper level of concrete-operational thought (stage IIIB) subjects drew perpendiculars within the figure (see fig. 5), such as BK or BK', to help them in their constructions. But at the stage of formal thought they preferred (although not invariably) to draw CK'', which is outside the triangle. This new construction does not indicate a new coordination different from that at IIIB, but it does argue a greater freedom from what is perceptually given and, as such, gives evidence of the early stages of formal-operational thought.

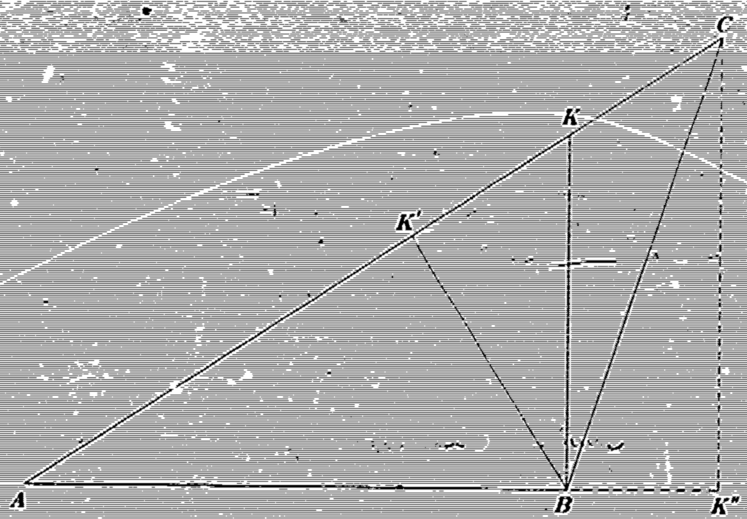

Fig. 5

Memory

In this third part of my talk I want to say a little about one of a number of tasks we have recently given to children which were based on the work of Piaget, Inhelder, and Sinclair (1968) dealing with memory and intelligence. I am not concerned here with memory and intelligence per se, but I want to suggest that children presented with mathematical, or other, ideas assimilate them to the schemes that they themselves possess. By the same token, the growth of understanding mathematical ideas is dependent on the growth of schemes, which in turn evolve through their own inherent functioning through the spontaneous experience and activity of the child.

Children aged five, six, and seven years were presented with a large bottle (about 1200 cc capacity).[2] It was half filled with coloured liquid

2. The study was carried out by Mrs. M. Caltieri and Mr. D. Nattriss.

and laid on its side in the prone position (see fig. 6). Placed alongside it on the table was a toy automobile with a horizontal line painted on it in order to emphasize horizontality. On the other side of the toy car was another bottle of the same size, half filled with coloured liquid and inverted. The diagram makes the arrangement clear. It will be recalled that Piaget and Inhelder (1956) showed that the notion of a Euclidean system of reference and that of horizontality is a relatively late acquisition, although our experience at Leeds is that while these ideas are acquired gradually, they come earlier and with greater individual differences in respect of age and situation than the Geneva school suggests.

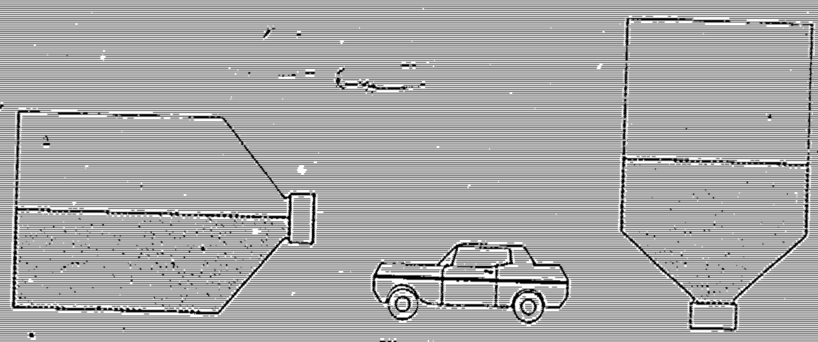

Fig. 6

After the children were shown the materials—and this was done with great care—the materials were removed and pupils were asked to draw the bottles and their contents. This was completed in late November and early December of 1969. The responses could be divided into the stages indicated below. It will be noticed that we devised stages for the drawing of each bottle and its contents, whereas Piaget, Inhelder, and Sinclair (1968) gave stages embracing the drawings of the two bottles and their contents.

Prone bottle

Stage 0: No recognisable bottle shape

Stage 1: Correct bottle shape, but entirely full or empty of liquid

Stage 2: Correct position of bottle, but liquid adhering to the base of the bottle, or to the top, or to both

Stage 3: Correct drawing

Upside-down bottle

Stage 0: No recognisable bottle shape

Stage 1: Bottle as in model, or reversed, but full or empty of liquid

Stage 2: Bottle the normal way up (i.e., the model was inverted) and liquid adhering to the top or side

Stage 3: Bottle drawn upside-down as in model, but liquid still adhering to the base or side

Stage 4*a*: Model inverted but liquid level shown correctly

Stage 4*b*: Bottle and liquid level correctly drawn

The numbers of children reaching these stages are given in tables 2 and 3.

TABLE 2

Prone Bottle

	5 yrs.	*6 yrs.*	*7 yrs.*
Stage 0	29	11	12
Stage 1	19	12	12
Stage 2	2	7	5
Stage 3	1	1	8
Total	51	31	37

TABLE 3

Upside-down Bottle

	5 yrs.	*6 yrs.*	*7 yrs.*
Stage 0	25	4	5
Stage 1	23	20	15
Stage 2	1	1	0
Stage 3	1	2	3
Stage 4*a*	0	4	6
Stage 4*b*	1	0	8
Total	51	31	37

The teachers in the schools were, of course, aware of the nature of the experiments we were carrying out. They were asked to do absolutely nothing to rehearse the experiment in any form or to engage, deliberately, in any teaching situation that would tend to remind the pupils of the models they drew. We have every reason to believe that the teachers observed our request.

At the end of May and the beginning of June of 1970 the same two investigators visited the same children and asked them to recall and draw the bottles and the liquids. The figures given in table 4 show the percentages of pupils showing regression, performance at the same level, or progression.

All the experiments we carried out broadly supported the Geneva position in respect to memory and intelligence. But this is not exactly the

TABLE 4

Age	Prone Bottle			Upside-down Bottle		
	regression	*same*	*progression*	*regression*	*same*	*progression*
5 years	26	65	9	7	50	43
6 years	23	37	40	10	53	37
7 years	29	35.5	35.5	6	46	48

way I want to put things here. What I want to say is that here is an example of a mathematical idea—the Euclidean frame of reference—that is developing through incidental experience and action and not through specific teaching. The development arises through the evolution of schemes. Note too that the upside-down bottle was an easier task than the prone bottle. I am in no sense belittling the value of good consistent teaching—indeed, no one would stress the value of such teaching more than I. But it is salutary to ponder on the fact that it appears that mathematical learning is dependent, in part, on general experience and action leading to the evolution of schemes, or as the Geneva school would say, such learning is dependent on the laws of development.

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HERMINE SINCLAIR

Piaget's Theory and Language Acquisition

It may seem rather contradictory that Genevan psychologists should be interested in the acquisition of language. Piaget himself has always stressed the fact that language is *structured* by thought rather than being its source, and since it is in the development of thought that he is interested, he has paid very little attention to language. Even if one of his first publications *The Language and Thought of the Child* (Piaget [a] 1923) had the word *language* in the title, this does not mean that at that time he accorded a more important status to language. He was then studying thought through the verbal interchange that takes place among children and between the child and an adult. In his later work, he found methods better suited to his purpose, although dialogue between child and experimenter always plays a part.

It was Piaget who dethroned language from the central position it had occupied in the minds of those who wanted to study thought, and it was Piaget who put language in its place by showing that it is part of a much more general capacity—that of representation or the symbolic functions, which appear in the middle of the second year in a number of different behaviors (symbolic play, delayed imitation in the absence of the model, mental images, etc.). However, he did not suppose that language acquisition would therefore follow the same line of development as the other manifestations of the symbolic function. He pointed out that language dealt with *signs*, that is to say, symbols that have no resemblance to, or other links with, the objects and events they symbolize.

On the other hand, because of the spectacular achievements of a linguist, Chomsky, language is now being reinstated as the key to the understanding of human thought. Furthermore, since it was shown conclusively that

language could not possibly be acquired in an associationist manner, it was concluded that the fundamental linguistic structures must therefore be innate. It may seem paradoxical in the light of these theoretical differences to say that it is Chomsky's work that is making possible the study of language acquisition within a Piagetian framework.

However, despite the important differences, Chomsky's and Piaget's theories have several points in common. Both men are nonempiricists, both are interested in underlying structures that can be formalized, both are dealing with competence rather than with performance. Also, it is through Chomsky's studies that the difference between language and the other manifestations of the symbolic function has become much clearer. In symbolic play and images, symbols may be linked in a common framework, but they do not form a system. Language, by contrast, is structured into a system, and although it is, on the one hand, a way of representing what is known, it is, on the other hand, itself an object to be known. The child has to infer regularities and rules and arrive at an *interiorized* grammar that will enable him to construct and understand an unlimited number of sentences in his mother tongue. It is in this sense that the study of language acquisition cannot be undertaken in the same manner as that of other modes of representation; but it is also in this sense that language is an object of knowledge and that its acquisition can be studied in the constructivist manner in which Piaget studies the development of other types of knowledge.

Theoretical Framework and Experimental Methods

Genevan research in language is based (as is most research) on certain theoretical principles. The basic ones are the following:

1. It is not language that explains human thought, but rather cognitive patterns and operations that will eventually provide the basis for explaining language.
2. It is not an explicit description of language as a fully acquired system that will throw light on acquisition mechanisms and provide a model for comprehension and production mechanisms, but rather a better knowledge of the long process of acquisition that will elucidate the functioning of the fully acquired system. This functioning of the system is not to be equated with a search for the psychological factors that distort "performance"; it concerns the mechanism through which *ideal* performance under optimal conditions is attained (in comprehension as well as in production), leaving aside the flaws caused by deficits in perception, memory, attention, and so on.

The above principles stem directly from Piaget's epistemological approach, of which the following are two of the aspects:

1. The development of thought is an autonomous process, but that of the symbolizing and representative functions is not.
2. The structure of adult thought can be understood only through the study of its formation, that is, the study of cognitive development in the child.

However, our guiding principles do not imply that we have a reductionist attitude toward language. We certainly do not think that language acquisition can be explained by the laws of cognitive development alone; the structure of language itself is a necessary part of an acquisition model.

This catalogue of basic convictions is no more than an explanation of our experimental methods. We are very far from having any coherent theories on acquisition, and our experimental studies are only just beginning. The most complete study to come out of the Geneva school is a doctoral thesis by Emilia Ferreiro (forthcoming) on the temporal relationships in children's language. Several projects are still being carried out, and it is not possible to give more than an indication of the implications we see in our results. However, before giving some examples, something more has to be said about our methods.

We try to investigate comprehension as well as production. We study comprehension in the following way: the experimenter pronounces a verbal pattern (such as "The boy is pushed by the girl"), and the child is asked to act it out using a number of toys. In many ways, this technique is superior to the one in which the subject is asked to choose among pictures. In several of our experiments the children understood the utterance in an unforseen way that we could not have represented pictorially. Some examples are given below.

In the production tasks we try to elicit a certain verbal pattern by asking the child to describe an event the experimenter has acted out with toys. Production is obviously very much more difficult to study than comprehension. In fact, since there are almost always several ways of describing an event, it is difficult to devise a situation that imposes a particular structuration. For example, how does one go about obtaining relative clauses from a child? In the case of adult subjects, one simply explains and asks for a particular pattern. But such methods are not possible below the age of eight or nine, before certain grammatical terms have been learned in school. We cannot, in general, decide from its nonappearance at certain ages that the children are incapable of producing the pattern in question. We are, however, less interested in success and failure than in types of errors and patterns. Evidently, we are not interested in

errors that are due to such superficial factors as lack of attention or of memory. We try to exclude them by keeping the child interested, by rephrasing the instructions, and, in the case of comprehension tasks, by repeating the utterance in question as often as seems necessary. Moreover, the appearance of a certain type of error or pattern at one point of development and not another, or that of a new type of error that seems to derive from the preceding one, cannot be explained by memory or attention factors. Such findings directly touch on the problem of determining the rules children's productions follow at different moments in development and how later rules are derived from earlier ones. In fact, when we speak of errors, this is not the right description for many of the verbal patterns we find. Often, the children's descriptions are quite adequate in themselves but different from adult expressions, and if grammatical errors are present, we can regard them as expressions of the type of child grammar our subjects work with, in which case they become precious indications of rule-bound behavior that simply follows different rules. The term *error* is applicable only when in the comprehension tasks children give a wrong interpretation of an unambiguous sentence.

In both comprehension and production tasks, we try to apply the Piagetian exploratory method; although we make sure that all the children are asked certain questions, we adapt the interview to each child's individual level. This exploratory method is much more difficult in language experiments than in others, one reason being that what Bärbel Inhelder calls *verification sur le vif* is often impossible. This *verification sur le vif* occurs when, on the basis of his knowledge of cognitive development, the experimenter makes a hypothesis on the thought pattern (scheme) that may underlie the child's answer. From there, the experimenter makes up a new question, a new situation, or a problem which will enable him to test his hypothesis. In other words, as soon as a subject reacts in an unforeseen way, the experimenter tries to find out what is behind the answer. Since we still know very little about an eventual sequence in the acquisition of particular patterns, it is difficult to invent new situations; since we cannot ask questions without using a certain verbal pattern, we may unwittingly influence the child's answer; and, finally, at the age at which we are working, children are incapable of understanding questions about language itself. In our language experiments we usually introduce an exploratory period at the end of the procedure; but we may also take advantage of children's spontaneous remarks such as "I think I got it upside down" and questions such as "Should I take the boy first?" With children who spontaneously start playing with our toys, we may change the order of our items to suit the framework of their play activity.

It is obvious that this kind of procedure leads to a certain loss in quan-

titative information. However, in our view this loss is more than compensated for by a corresponding gain in qualitative information.

In certain cases, the children are also asked to repeat something the experimenter has said, in a context where they have to make an effort to understand the utterance. For instance, from time to time during a comprehension task, we ask the child to repeat what we have said before he acts it out with the toys. Interestingly, as can be seen in the examples below, these repetitions often resemble the answers we obtain in production tasks. They also may throw light on the peculiar reactions that occur during the comprehension task. A comparison of reactions to all three tasks is therefore often fruitful. Incidentally, the repetition task provides a kind of check on memory factors.

Before giving some examples of Genevan research in language, I should like to bring up a question that is often asked: How is it possible to hope to find a parallel between the mechanisms of cognitive development (as they have been interpreted by Piaget) and those of language acquisition if the latter seem to take place in a continuous, direct manner without any discernible stages and, especially, if the main linguistic structures are all present at about the age of five, as is often maintained?

There is no language in the proper sense before the culmination of the sensorimotor period. Communication, however, starts right from birth (cries of distress and simple signalling by the baby of its presence in a certain place). Communication soon becomes bipolar: distress is signalled by generally high, nasalized sounds, produced with tense muscles; contentment is signalled by low, nonnasal sounds, produced with a relaxed musculature.

Little by little, vocalisations take on some of the phonetical and prosodical characteristics of the mother tongue, under the influence of many different factors (such as the development of recognition-memory, adoption of a sitting position which allows the child to look directly at another person's face, the beginnings of imitation in the presence of a model, sensorimotor coördinations between sight and hearing, development of the muscular tonus in the lips, etc.).

With the acquisition of object permanency and the first grouplike structure, the infant becomes able, in our view, to start communicating with language. In fact, object permanence means that objects are now becoming things to be known and not only things to be reacted to. This development at the same time *necessitates* some kind of representation and *makes it possible*. Language is, among other things, a capacity to substitute a signifier for real events and objects. It is therefore comprehensible that these realities have first to be established as having an existence divorced from the subject's actions before they can be repre-

sented by a signifier. At this point, Piaget's stages and periods of language acquisition converge. As I have said, language proper starts at the end of the sensorimotor period and coincides with the advent of other representational behavior.

One characteristic that distinguishes the first verbal period from what went before is that instead of a direct communication between interlocutors (which continues to exist, of course, even between adults—e.g., expressions such as *sssh* or *brrr*), communication now includes a reference to some event or object signified in an as yet very global but already conventional manner, in sounds that the adult recognizes as a word (or a combination of several words) from his language.

Evidently, the very first manifestations of language, the famous first words, are difficult to place exactly. As with all imitative behaviors, they have to be recognized as such by the observer. For example, if I had not seen a seventeen-month-old girl look intently at a butterfly beating its wings against a window, I would not have interpreted her behavior, on a visit a week later, when she went straight to that window and flapped her arms, as a delayed imitation of a past event. Also, there is a great danger of interpreting the first words in an adultomorphic way (e.g., interpreting them as nouns, or verbs) and parochial manner (e.g., fitting them forcibly into the word classes of a certain language). Without going into the controversy of the precise nature of this holophrastic speech, I should like to allude to Piaget (1959) who considers holophrases to be *l'enounce d'une action possible* ("expressing a possible action") and *jugement d'action* ("action judgment"). In other words, he emphasizes their predicative function; that is, they say something *about* something.

A second characteristic of the first verbal productions is the discovery of the principle of syntactic combination. First, sometimes in the form of a simple juxtaposition of two separate elements, then in the expression of genuine relations ("daddy car"), syntax begins to develop. The two- or three-word phrases—such as "read book", "that cat fish", "no come eat", "Johnny truck"—announce grammatical relations such as attribution, possession, localisation, and so on. The first vocalisations seem to be phonetically universal (all babies produce the same sounds, whatever the language spoken in their environment); the first holophrastic or two-word utterances seem to be structurally universal. But very soon, with the introduction of the first grammatical markers, the particular system of the mother tongue appears.

It has been amply demonstrated that although imitation obviously plays a role, these first productions are not shortened and clumsy copies of adult sentences but are active creations on the part of the child. An older child's saying "I'll smile up my painting" (putting a smiling mouth

in the face of the sun) is a quasi-poetical extension of a semantic field.

By the age of five, it seems that the child has acquired most of the adult's syntactic structurations. If this were true, all language acquisition would take place during Piaget's preoperational period. The precocity of this achievement is often taken as an argument for the innateness of linguistic structures. However, in the first plac[illegible] as Hans Furth (1969) has said, "There is no reason to believe that ot[illegible] knowing and symbolic behavior that the child acquires as early as language does not require equally complex activities on the part of the child". Indeed, the recent work of Piaget et al. (1968) on the preoperational period with its semi-logic of unidirectional dependencies has made the achievements of this stage much clearer. Moreover, many of the typical conquests of the concrete-operational period are already present, in isolated thought patterns, much earlier. The four-year-old *knows*, and will say so if asked, that in the liquid-pouring experiment nothing has been added or taken away; he perceives that the glasses have different dimensions; he knows by that age that if you pour the liquid back into the original glass, it will attain the original level (*renversabilité*). But what he cannot do is to derive from all this that the quantity of the liquid must therefore be the same. This conclusion will become logically necessary only with the establishment of the operational structure. In the second place, many linguistic structurations are not understood until the age of nine or ten; the acquisition period of a number of fundamental and frequent patterns stretches well into the concrete-operational stage.

Our hypothesis is that the linguistic competence of children of four to five years of age has the same functional characteristics as their cognitive competence and that it does not necessitate the intervention of concrete operations. This hypothesis does not imply that the one-way mappings of this period *explain* the verbal acquisitions; in fact, one of the experiments, which I shall discuss later, shows clearly that many problems remain to be solved.

In what way, then, does the period of concrete operations elaborate language acquisition? It would seem that several new capacities start to lead to new verbal behavior at this level.

In the first place, a certain reflexion on language itself becomes possible, since sentences can now be dissociated from their content. When asked to make a sentence with the words "coffee" and "salt", a six-year-old will no longer say, "You can't, nobody puts salt in coffee!" He will be able to answer questions on sentences such as "How many words do I say when I say, 'Mary has seven dolls'?" The four-year-old will announce, "Seven words"; the six- to seven-year-old will count on his fingers and get the right answer (or the wrong answer, but it won't be seven).

In the second place, this detachment will make it possible for him to find different formulations describing the same event. It now becomes possible to *conserve* the semantic content of an utterance while changing its form. This may sound like a facile allusion to conservation concepts, but although I believe that it has a more profound significance, I cannot as yet explain it further. For instance, in our own experiment on the passive sentence (Sinclair and Ferreiro 1971), five-year-olds often behaved as follows: when the experimenter had shown them a truck pushing a car and asked them to describe this event by starting to speak about the car, they said, "No, I can't; otherwise it would be the wrong way round, the car pushing the truck". Similarly, in Ferreiro's experiment on temporal relationships, when five-year-olds were asked to describe two events acted out, such as a girl doll going up a staircase and then a boy doll entering a garage, they rigidly adhered to this temporal succession in their description. They either refused to start with "the boy" or if they tried to do so, they were incapable of introducing temporal indicators to reestablish the correct order. They finished by admitting that when one starts with the second event, "It's the wrong way round", or, straightforwardly, "It's wrong" and claiming that it could not be done. Younger children (four-year-olds) did not express these perplexities. They simply complied with our instructions and inversed agent and patient ("The car pushes the truck" instead of "The car is pushed by the truck") and also reversed the order of the two events without hesitation (maybe they expected reality to be adjusted to their description).

For the moment, we cannot yet go beyond a description of such new acquisitions that take place during the concrete-operational period. However, from our few experiments we are convinced that these acquisitions concern not only lexical but also syntactic acquisitions. We hypothesize that the new structure of thought makes these acquisitions possible and that it will be possible to show parallels. The operational period would in this view constitute a restructuration of the acquisitions in the preoperational period.

Two Experiments

After this long introduction, I shall briefly indicate some of our findings from two experiments on passive sentences and on the ordering of three words that are not linked by a syntactic structure. The technique for the study of passive sentences has already been described: comprehension, through the acting out of passive sentences said by the experimenter; production, following the instruction to describe an event acted out by the experimenter by first naming the patient; and repetition of

the sentence said by the experimenter before acting it out. In general—and we conducted the experiment with both French- and German-speaking children—our results are the same as those found by Harry Beilin (1969) with subjects of English mother tongue.

Our criterion for success in the comprehension task was the correct acting out of what Slobin calls *reversible sentences*, such as "The boy was washed by the girl", in contrast with such sentences as "The car was washed by the boy". In the first example, agent and patient can be interchanged without rendering the semantic content of the sentence impossible or improbable. Girls can wash boys and boys can wash girls. In the other example, only one version is possible, since cars don't wash boys!

Comprehension results indicated that by six years and six months to seven years, most of these sentences are understood. Before then, the main tendency is to take the first noun as the agent and the second as the patient. The five-year-olds hesitated a great deal, and there were some interesting compromise solutions. For instance, some of them introduced a kind of reciprocal action, and for the sentence "The boy is washed by the girl", they made the boy wash the girl *and* the girl wash the boy. Interestingly, apart from the difference between reversible and irreversible sentences, the particular action presented (or the verb chosen) influences both the average age at which success is achieved and the type of utterance produced. A durative verb such as *laver* ("to wash") is not treated in the same way as verbs such as *renverser* ("to knock down"). Passive sentences with the verb *suivre* ("to follow") are often not understood by children of eight and even beyond that age.

The repetition task showed that at four years of age children are perfectly capable of repeating passive sentences correctly, but their comprehension is incorrect. As mentioned above, they act as if the subject of the passive sentence was the agent. They also introduce certain modifications in the model that are difficult to interpret. These modifications, however, concern either the verb form (*a lavé*, "has washed", instead of *est lavé*, "is washed", for instance) or the preposition (*avec*, "with", instead of *par*, "by") or both, but never nouns.

At the age of five, comprehension gets better, but repetition seems to deteriorate. In fact, this apparent deterioration is an improvement; parrot-like correct repetitions disappear, and different modifications are introduced. The most interesting are those that transform the passive sentence into an active one, keeping the semantic content constant ("The boy is washed by the girl" repeated as "The girl washes the boy") or those that substitute another verb form for the passive ("The boy got washed by the girl"). Only from six years of age are literally correct repetitions always accompanied by correct actions.

The production task showed that our instruction (to start with the patient instead of the agent) is interpreted by the four-year-olds as a suggestion to "say something about" the patient, which in many cases leads to a truncated passive or an intransitive sentence that does not mention the agent; for example, "The boy gets clean", "The boy is washed", "The car moves", and so on. The five-year-olds are capable of centering their attention on both the patient and the agent, but they do this in two juxtaposed sentences, of which one describes the patient and the other the action; for example, "The boy is clean and the girl washed him". Around the age of six, the first passive constructions are regularly used, although (in French) our children prefer the formula *il s'est fait laver par la fille* (which can be translated roughly as "he got himself washed by the girl").

In this experiment, the results of the comprehension, production, and repetition tasks converge. They also reveal a phenomenon of decentration, which is one of the characteristics of the concrete-operational period.

The other experiment is much more difficult to interpret. Having observed that word order is an important semantic indicator for young children, we wondered how they would interpret a sequence of three lexical items not linked by a syntactic structure and, especially, whether the order in which the three items were presented would have some influence on their interpretations. We chose three combinations:

1. Two nouns and one transitive verb, such as boy-girl-wash or boy-box-open (reversible and nonreversible), presented in the six possible orders (boy-girl-wash; girl-boy-wash; girl-wash-boy; boy-wash-girl; wash-boy-girl; wash-girl-boy)
2. Two nouns and one intransitive verb, such as horse-pig-run
3. Two verbs and one noun, such as bear-jump-grunt

In this experiment, our youngest subjects were three years old. We warned the children that we were going to "talk rather funnily, not at all as the teacher would talk" and that they should try to "find out what we meant and show us what they thought". Almost all the subjects accepted this instruction and only a few three-year-olds did not seem to be able to do anything but choose the toys named in the combination, without making them act.

A first surprise was that there was a definite difference in the children's behavior according to whether a transitive or an intransitive verb was used. The latter did not give rise to any hesitations. The children made the horse and the pig jump, or the bear (provided with a grunting mechanism) jump and grunt, often in the order in which we had pronounced the items, but often not. On the other hand, the combinations

with a transitive verb gave rise to hesitation and periods of reflexion. It seemed as if as early as the age of three, there is some awareness of the difference between transitive and intransitive verbs, and moreover, the children behaved as if in the case of a transitive verb, word order becomes important.

Since the experiment is not yet finished, I can do no more than give some indications of what happens with transitive verbs. A definite change does seem to take place between four and seven.

The nonreversible combinations (such as boy-wash-car) are at all ages preponderantly interpreted in the most probable way, in whichever order they were presented. However, a few five-year-olds, in the case of the order car-wash-boy, put our little sponge on top of the car and made the car wash the boy. This very strong conviction that "the first noun is the subject" is also attested to by the fact that in the reversible combinations (such as boy-wash-girl) all the children, of all ages, took the first word as the agent. However, the age differences appeared when the order was not noun-verb-noun but either noun-noun-verb or verb-noun-noun.

The youngest subjects showed a strong tendency in the case of verb-noun-noun to introduce an agent from outside: for instance, when we said "wash-boy-girl", they themselves took the sponge and washed both dolls or took a third doll and made her wash the others. Sometimes, in the case of noun-noun-verb they introduced a patient. This behavior again seems to indicate a very strong awareness of the subject-verb-object constiuction; if a possible subject in the form of a noun is not present as the first word pronounced, they introduce one.

The slightly older subjects (four and five) had different solutions to this problem. Sometimes, we again observed the reciprocal solutions, as in the passive sentences. For "boy-girl-wash", the children made the boy and girl wash each other. Two other solutions seem to become preponderant around the age of five. The first is to consider the noun immediately following the verb as its object. For example, "wash-boy-girl" is interpreted as "The girl washes the boy". The other solution is to regard the first of the two nouns as the subject and the other as the object. Thus, "wash-boy-girl" is understood as "The boy washes the girl". The last solution seems to be preferred at all later ages. In the first solution it is the verb that is picked out by the child and interpreted in terms of its position in relation to the nouns; in the other, it seems to be the order of the two nouns that is decisive for the subject-object relationship.

These first results seem to us rather more mysterious than those of the experiment on the passive sentences. In view of the predominance of ordinal relationships during the preoperational period, the early aware-

ness of order for the interpretation of utterances may not be so surprising. However, the systematic nature of our young subjects' interpretations was astonishing. Moreover, the differentiation between transitive and intransitive verbs and the shift from a verb-centered to a noun-centered interpretation is difficult to understand. We conducted this experiment in a rigid way and have not yet introduced any supplementary questions or situations to explore the mechanism behind the behavior. As always, we hope that the children's own explanations will give us some more ideas.

In conclusion, it can only be said that the mystery of language acquisition is far from solved. Obviously, the small number of experiments performed in Geneva and elsewhere cannot possibly lead us immediately to a comprehensive theory of acquisition mechanisms, which could then be tested against new facts. However, it would seem that by taking into account the laws of cognitive development we shall be able to get some idea of the kind of inferences children at different stages of development are capable of making from the adult utterances they hear. No theory of acquisition can afford to ignore the important work done in linguistic theory; but neither can such theories ignore the stages in cognitive development, which, in our opinion, will give the clue to the nature and form of the linguistic structures children are able to produce and understand.

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MILLIE ALMY

Longitudinal Studies Related to the Classroom

The decade just ended, marked at its beginning in the United States by J. Bruner's book *The Process of Education* and J. McV. Hunt's *Experience and Education,* and in England by K. Lovell's *The Growth of Basic Mathematical and Scientific Concepts in Children,* saw a tremendous upswing in interest in the work of Jean Piaget. Many of his experiments have been replicated. Many researchers have attempted to train children in the acquisition of the concepts he has studied. The developers of the many new curricula seem to have felt bound to cite Piaget, although the number of programs directly based on his theory, at least in this country, have been few.

Reviewing the Piaget-inspired research of the decade, I have been impressed with the scarcity of that taking a truly developmental, or longitudinal, approach and also with the scarcity of that having direct relevance to the classroom. The studies to be reported here represent an effort to take a longitudinal view of classroom innovation.

Both of these studies have now been published by Teachers College Press, the first as *Young Children's Thinking* in 1966, the second as *Logical Thinking in Second Grade* in 1970. I shall sketch only enough of what we did, and the results we obtained, to enable you to understand their longitudinal nature and classroom orientation. Then I shall discuss some theoretical and methodological problems encountered in longitudinal research as well as its promise.

The first study, initiated in the fall of 1961, was concerned with the extent of the ability to conserve among young children in two metropolitan elementary schools, one in a middle-class neighborhood, the other in a lower-class neighborhood with a mixed population identified as Puerto Rican, Negro, and other. Only those children whose language ability in English was deemed adequate by both their teacher and our interviewers were included in the study. Individual interviews following standardized procedures were used to determine the children's ability to conserve. The content of the interviews is schematized in figures 1 and 2. Task A deals with the conservation of the equality of number of two sets of blocks through two transformations, Task B with the identity of number of one set through two transformations following an opportunity to count, and Task C with an amount of liquid after a single transformation. The orientation sections attempted to focus the child's attention on number and quantity, and we used the "What about now?" question to secure an answer without giving the child further clues. Where necessary, we asked more specific questions: "Is there more here? Or more there? Or do we have the same?" But no probing was done beyond the request for an explanation: "Why do you think so?"

Our scoring procedures were designed to take into account the nature of the evidence regarding the child's ability to conserve—for example, whether or not he responded spontaneously to the "What about now?" or required more specific questions. Eventually we were able to categorize each child's performance as falling in one of four patterns: conservation in no task; conservation only after counting; conservation in that task and in the other block task, but not in the amount of liquid task; conservation in all three tasks.

In this first study we had about fifty middle-class and thirty lower-class children at each grade, from kindergarten through second grade. The percent of children who were able to conserve in all three tasks shifted from 9 percent in kindergarten to 48 percent in second grade in the middle-class school. In the lower-class school only one child conserved in all three tasks in kindergarten, and only 23 percent by second grade.

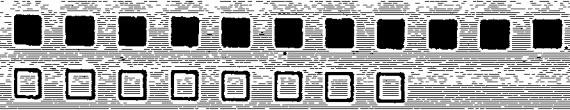
The apparent orderliness of acquisition of the ability to conserve discontinuous and continuous quantity confirmed the research of the Genevans, while the discrepancy between the two schools raised a number of interesting questions.

The longitudinal part of this study followed the kindergarten children into the second grade. Forty-one of the middle-class group, and twenty-four of the lower-class group were reinterviewed, using the same schedule, in the spring of the kindergarten year, in the fall and spring of first grade, and again in the fall of second grade—at approximately the same time as

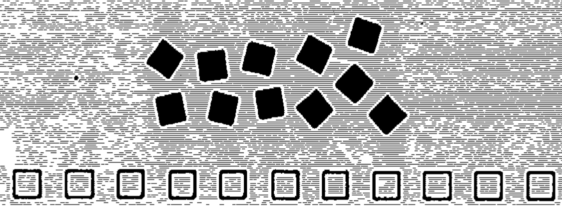
Figure 1

CONSERVATION OF NUMBER

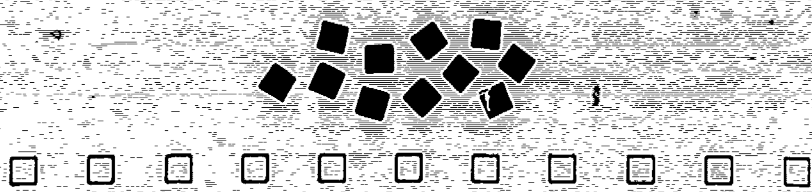
Task A: Orientation



ARE THERE JUST AS MANY YELLOW BLOCKS AS RED BLOCKS? YOU TAKE SOME MORE AND MAKE IT SO THERE ARE JUST AS MANY. After child does so (or is assisted in doing so), E removes 1 red, asks WHAT ABOUT NOW? and, if necessary, ARE THERE JUST AS MANY RED ONES AS YELLOW ONES? ARE THERE MORE YELLOW ONES? Continues removing 2 yellows, returning 2 yellows, etc. ending with child's statement of equality.



WHAT ABOUT NOW?



WHAT ABOUT NOW?

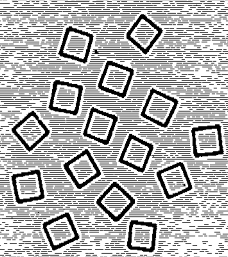
WHY DO YOU THINK SO?

Fig. 1

Task B: Counting

CAN YOU COUNT? CAN YOU FIND OUT HOW MANY BLOCKS THERE ARE? YOU COUNT THEM. Assist as necessary. SO HOW MANY BLOCKS ARE THERE?

HOW MANY NOW? (CAN YOU TELL WITHOUT COUNTING?)



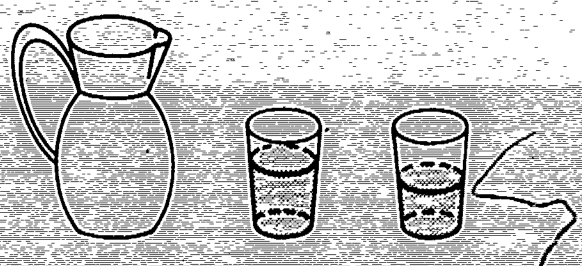
HOW MANY NOW?

Fig. 1—*Cont.*

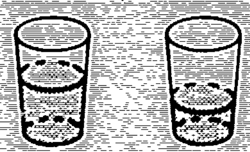
Figure 2

CONSERVATION OF A QUANTITY OF LIQUID

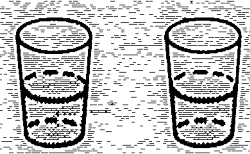
Task C: Orientation



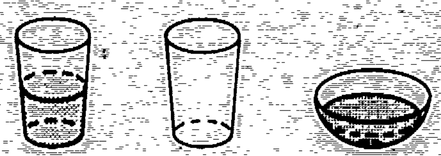
WHICH HAS MORE? CAN YOU MAKE THEM SO THEY ARE THE SAME? When child has established equality, E returns some of the water from one glass to pitcher.



WHAT ABOUT NOW?



WHAT ABOUT NOW?



WHAT ABOUT NOW? WHY DO YOU THINK SO?

Fig. 2

that when the second graders in the first part of the study had been interviewed.

The longitudinal results confirmed those of the cross-sectional study, as far as the sequence of acquisition of these conservation abilities and the differences between the two groups were concerned. They also enabled us to examine the relationships between progress in conservation and functioning in other areas of the curriculum, such as beginning reading and mathematics.

One finding of the longitudinal study was that a larger percent of the middle-class children were conserving in all three tasks when they reached second grade than had been the case for the second graders in the cross-sectional study. Whether the difference was attributable to repeated testing, or to somewhat better mental ability in the longitudinal group, or to innovations in the mathematics curriculum, or (more likely) to an interaction among these factors, could not be determined. It did, however, stimulate our interest in the extent to which the so-called new mathematics programs might affect children's ability to conserve and other aspects of the development of logical thinking.

We were also impressed by the extent of individual variation in the way progress toward conservation appeared to be made. While the general sequence of acquisition held for all the children, about half of them, in at least one of the interviews, regressed from what had appeared to be an understanding of conservation at an earlier period. Piaget attributed this finding to our standardized procedure and noted the importance of determining whether the response from which the child might have seemed to regress was not in fact indicative of a transition. In this case the child's expressed conviction should be regarded as tentative rather than certain.

The next study developed out of our interest in the effects of the curriculum on children's progress in logical thinking. It also throws some light on the question of individual variability, although it was not designed to do so.

The plan for this study evolved from a combination of circumstances that made it possible to compare the thinking in second grade of two groups of children—children who from kindergarten had received systematic instruction in the basic concepts of science and mathematics and children who had not had such instruction.

While the first study was in progress, I served as a consultant to the Science Curriculum Improvement Study (SCIS). This project was concerned with the development of scientific literacy through the provision of experiences of a scientific kind beginning in the early elementary school. It drew on the theory of Piaget, particularly in its emphasis on the importance of the child's discovery or construction of the concepts

presented to him. In the first two summers of the project when the first trial programs for kindergarten and first grade were being formulated, I interviewed many of the children informally, using modified Piagetian techniques, to assess the level of understanding they brought to the experimental program and tried to observe how they functioned in it. In the fall of 1965 the program began in the kindergartens and first grades of three California schools.

Somewhat prior to the development of the SCIS program, the American Association for the Advancement of Science (AAAS) had also begun to develop a science program starting in kindergarten. The two programs were similar in many respects. Both were parts of a complete and integrated elementary-science program formulated by men who were themselves scientists. The SCIS program emphasized concepts and the child's involvement with phenomena in the world around him. It assumed that the child will learn the processes of science as he experiments, discusses, and analyzes. In contrast, AAAS put the practice of the processes in the foreground, using the phenomena as a means for learning the processes, and the concepts as tools for understanding them.

The processes included in the AAAS program were observing, recognizing and using number relations, measuring, recognizing and using space-time relations, classifying, communicating, inferring, and predicting. In the kindergarten–first-grade program SCIS emphasized the concepts of physical objects and their properties.

A content analysis of the teacher's manuals provided for the two programs indicated some differences in the cognitive processes called for. For SCIS, using Piaget's terminology, 70 percent were concerned with the logic of classes, with an emphasis on grouping and describing. Approximately 10 percent dealt with the logic of relations. In the AAAS program approximately half pertained to the logic of classes, 15 percent to the logic of relations, and 20 percent to number. Both programs gave some attention to having the child become aware of his own thought processes through exploring, predicting, checking, and changing criteria for grouping. Both involved some opportunity for the direct manipulation of physical objects, with SCIS placing somewhat greater emphasis on this. There is of course no guarantee that the classroom experience of the children actually reflects these emphases. It is, however, possible to analyze the verbal interaction of the classroom to determine this.

Both these programs provided considerable contrast to the unsystematic science experiences usually provided in kindergarten and first grade. It seemed possible that more than usual experience in classifying and, to a lesser extent, ordering might result in earlier conservation, or it might in some way modify the patterns that had appeared in the earlier study. The

total scope of this study is indicated in the table on the opposite page.

Table 1 shows the number of children who were in the AAAS and SCIS programs at the beginning of the study in 1965 and who remained in them in subsequent years. Note that these children also had the Greater Cleveland Mathematics Program (GCMP) and that a comparable group of children had only the GCMP program, that is, no science program.

The program marked "No prescribed lessons" was selected to provide a contrast to the programs where the lesson plans were specified in considerable detail. The curriculum guides for this program, which had been prepared by committees of teachers and supervisors and were intended to meet local needs, recommended the general topics to be covered in kindergarten and first grades but did not include any detailed prescriptions for teaching such topics.

In the fall of 1966 another group of children were added who had come from a kindergarten program where they had not received instruction in either mathematics or science. Half of these classes participated in the SCIS first-grade program. All of them received instruction in mathematics, beginning in first grade. No one textbook was used, but the program can be regarded as comparable to the Greater Cleveland Program.

The major question raised in this study was: Do children who receive systematic instruction in the basic concepts of mathematics and science when they are in the kindergarten think more logically in Piaget's terms when they reach second grade than do children who did not have such early instruction? Obviously, an adequate answer to this question requires that the samples of children representing the various programs be comparable and that the teaching in each of the programs be congruent with its aims and of quality comparable to that in the other programs.

The published report (Almy et al. 1970) goes into these details, describing the intellectual ability of the children as measured by the Peabody Picture Vocabulary test, the occupational level of their parents, and the teaching as observed by experts representing the instructional programs and reported by the teachers themselves. As might be anticipated, the evidence for comparability of the groups is better than the evidence about the teaching. Accordingly, the answer to the major question remains somewhat equivocal. However, the picture of the thinking of second graders that emerges from our interviews with the 633 who remained in the study to its completion is not equivocal and constitutes a major contribution of the study.

Each child was interviewed either two or three times, depending on whether he was in a program initiated in kindergarten or in one that did not start until first grade. Each of these interviews presented the same

TABLE 1

Experimental Programs, Number of Schools, Classes and Children, 1965-1967

Program	Schools	1965-1966 Classes	1965-1966 Children	1966-1967 Classes[b]	1966-1967 Children	Fall, 1968 Classes[c]	Fall, 1968 Children
		Program Initiated in Kindergarten					
AAAS (GCMP)	5	7	189	8	105	11	94
SCIS (GCMP)	3	6	159	10	118	13	79
GCMP only	4	7	168	17	143	16	122
No Prescribed Lessons	4	16	181[a]	11	152	14	136
		Program Initiated in First Grade					
SCIS (Math)	3	-	-	8	139	15	115
Math only	2	-	-	8	113	14	87

[a]Does not include children in these classes enrolled in public kindergarten, but scheduled for first grade in parochial school.

[b]Classes include children who had not participated in program in kindergarten.

[c]Classes include children who had not participated in program in kindergarten and/or first grade.

conservation tasks that were used in the first and second studies, described earlier. In addition, we included a class-inclusion task as schematized in figure 3. Procedures for this task had been developed for a training study conducted by Paula Miller (1966), so that normative data for kindergartners and second graders were available.

The performance of the children in the successive administrations of the conservation and class-inclusion tasks constitutes the longitudinal data for the study. However, in the final, or posttest, interview five additional kinds of tasks were added.

Our intent was to provide a battery of tests from which some kind of index of logical or operational thinking might be derived. The criteria for the selection of the tasks are described in detail in the published report, but it is well to note here that the procedures were standardized, that they had all been used in relatively comparable form by researchers other than Piaget, and that normative data on the tasks were available.

Bear in mind that in the spring of 1968 when this battery was developed, very little testing of the same children with a variety of Piaget tasks had been reported. Nor had there been much of the kind of exchange between Genevan and American researchers that is currently going on. If we were designing such a battery of tests today, we might select a somewhat different array and use somewhat different procedures.

A set of serial ordering tasks were posed, with two sets of cards picturing monkeys and balloons and two sets picturing knives and forks. A schematization of the task appears in figure 4 (pp. 227–28).

We tested the child's ability to grasp the idea of transitivity by asking him to deduce the relationship of two black sticks each of which had been measured against the same blue stick. In each instance the black sticks are presented in the context of the Muller-Lyer illusion, tending to mislead the child who relies on perceptual cues. (One child described this as an "obstacle illusion.") Figure 5 schematizes the tasks.

A set of matrix tasks, taken directly from Piaget, were presented next. Figure 6 shows the cards used for these. However, it is difficult to tell when a child may be using a graphic solution and when an operational solution. In view of this the Genevans, as we learned from Dr. Sinclair, regard the matrix as one of the least good tasks for appraising operational thinking. Accordingly, we treated the results from these tasks quite separately from the results in the other tasks.

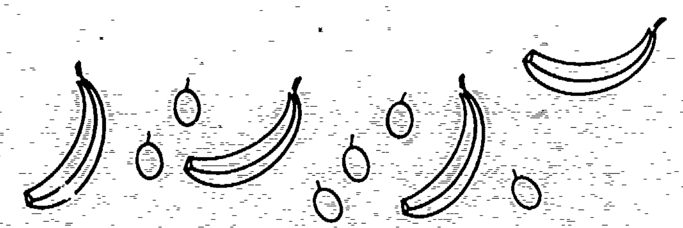
The final task in the posttest interview is schematized in figure 7 (p. 232). It deals with the conservation of weight.

The categorization and scoring procedures in this study were similar to those used in the previous studies. Essentially they consisted in weighing the evidence as to whether or not the child was thinking at an operational

Figure 3

CLASS INCLUSION TASKS

Task A: Fruit (4 plastic bananas, 6 plastic grapes)

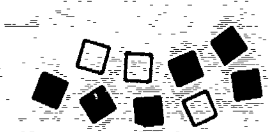


Orientation: HOW ARE ALL THESE OBJECTS ALIKE? WHAT DO YOU CALL THEM? CAN YOU FIND SOME WAY TO PUT THESE OBJECTS INTO TWO GROUPS WHICH BELONG TOGETHER? PUT ALL THE FRUIT INTO ONE GROUP.

SUPPOSE I WANTED ALL THE GRAPES AND YOU WANTED ALL THE FRUIT. WHO WOULD HAVE MORE PIECES OF FRUIT?

HOW CAN YOU TELL?

Task B: Wooden Blocks (6 blue and 3 orange)



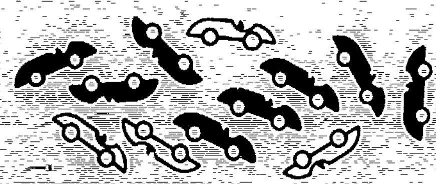
Orientation: CAN YOU FIND SOME WAY TO PUT THESE OBJECTS INTO TWO GROUPS WHICH BELONG TOGETHER? PUT ALL OF THE WOODEN BLOCKS INTO ONE GROUP.

WOULD A TOWER MADE OUT OF ALL THE WOODEN BLOCKS BE TALLER OR SHORTER THAN A TOWER MADE OUT OF ALL THE BLUE BLOCKS?

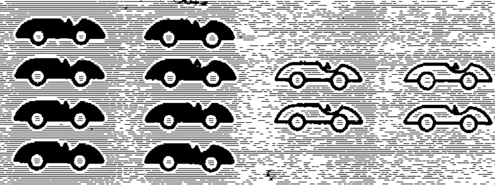
HOW CAN YOU TELL?

Fig. 3

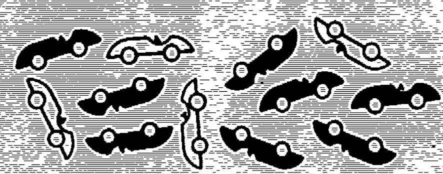
Task C: Metal Cars (8 blue and 4 red)



Orientation: HOW ARE ALL OF THESE ALIKE? WHAT DO YOU CALL THEM? CAN YOU FIND SOME WAY TO PUT THESE OBJECTS INTO GROUPS WHICH BELONG TOGETHER?



PUT ALL THE METAL CARS INTO ONE GROUP.

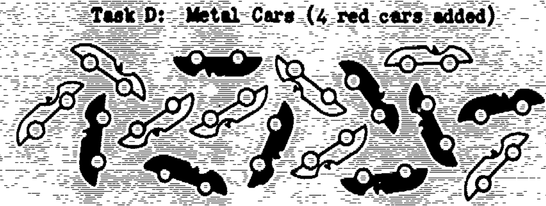


ARE THERE MORE RED CARS, MORE BLUE CARS OR ARE THEY THE SAME? HOW CAN YOU TELL?

ARE THERE MORE BLUE CARS, MORE METAL CARS OR ARE THEY THE SAME?

HOW CAN YOU TELL?

Task D: Metal Cars (4 red cars added)



ARE THERE MORE RED CARS, MORE METAL CARS OR ARE THEY THE SAME?

HOW CAN YOU TELL?

Fig. 3—*Cont.*

Figure 4

SERIATION

I. Monkeys and Balloons

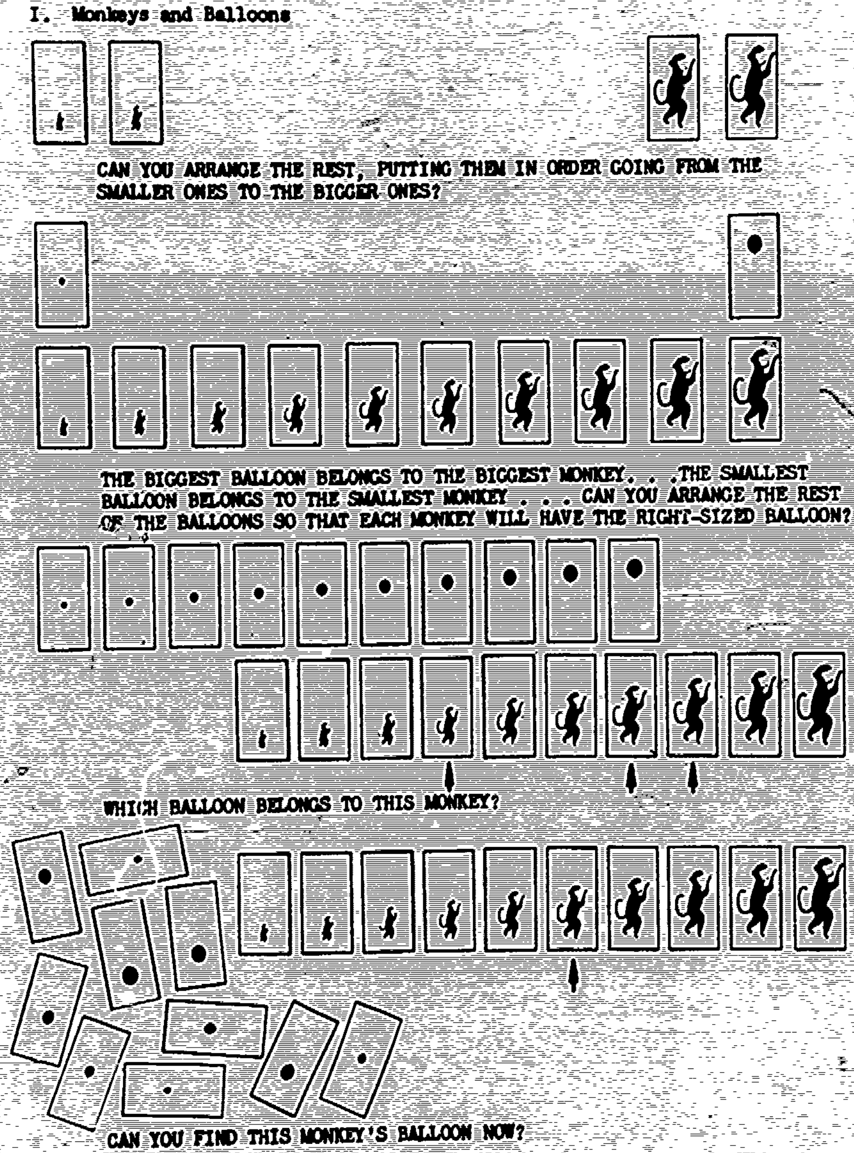
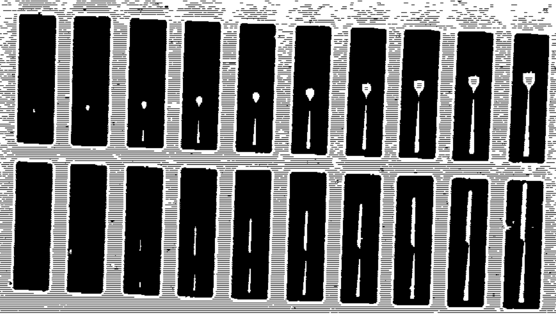
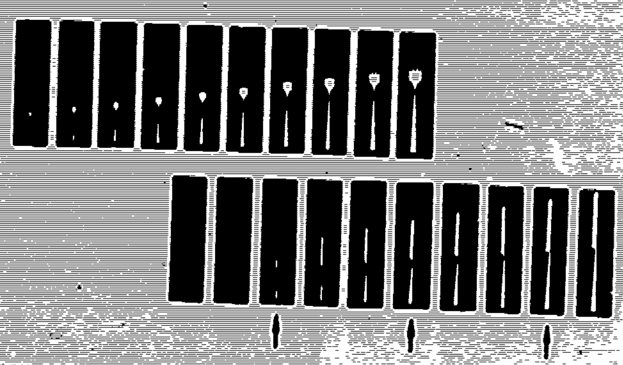


Fig. 4

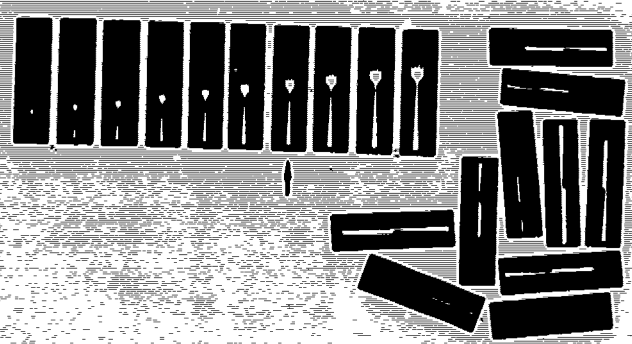
II. Knives and Forks



HERE ARE SOME KNIVES . . .IN ORDER, FROM LARGEST TO SMALLEST . . . HERE ARE SOME FORKS . . . I'LL PUT EACH FORK ABOVE THE KNIFE IT GOES WITH.



WHICH FORK GOES WITH THIS KNIFE?



CAN YOU FIND THE KNIFE THAT GOES WITH THIS FORK?

Fig. 4—*Cont.*

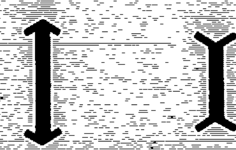
Figure 5

TRANSITIVITY

Practice I

WHICH ONE OF THESE TWO STICKS IS LONGER? DON'T COUNT THE ARROWS. JUST LOOK AT THE STICKS.

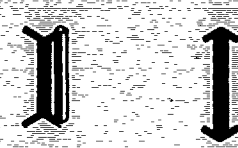
Practice II



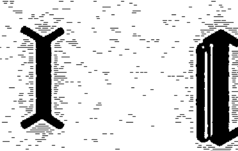
WHICH ONE OF THESE TWO STICKS IS LONGER? REMEMBER NOT TO COUNT THE ARROWS, JUST THE STICKS.

Test A.

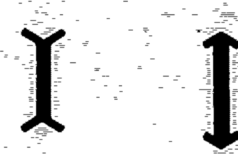
BEFORE YOU TELL ME WHICH IS LONGER I WILL PLACE THE BLUE STICK LIKE THIS.



WHICH IS LONGER, THE BLUE OR THE BLACK?



WHICH IS LONGER, THE BLUE OR THE BLACK?



WHICH OF THE BLACK STICKS IS LONGER?

Fig. 5

Figure 6*

MATRICES

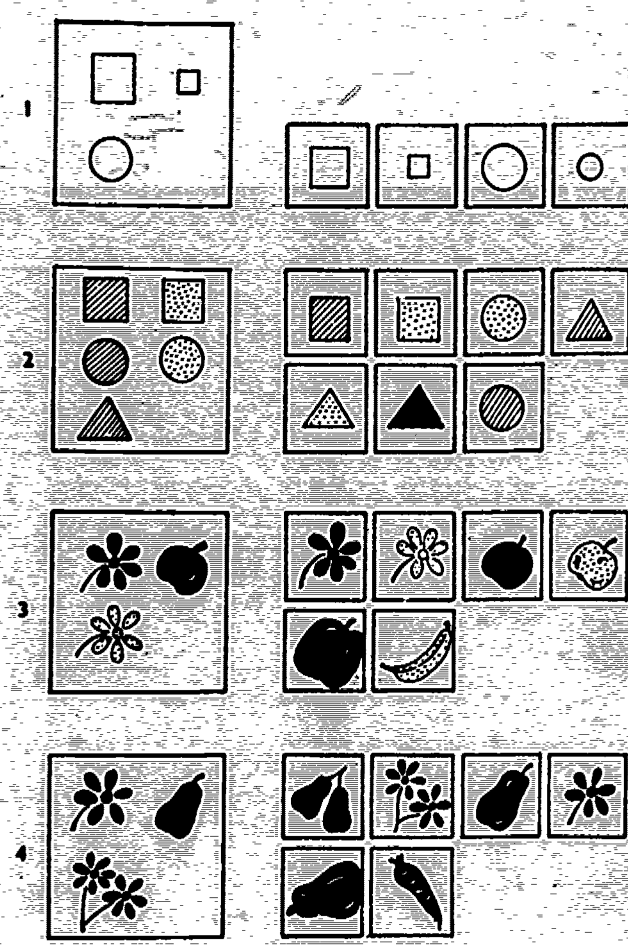


Fig. 6

*Fig. 10, pp. 160–61 from *The Early Growth of Logic in the Child* by Barbel Inhelder and Jean Piaget, translated by E. A. Lunzer and D. Papert. Copyright © 1964 in the English translation by Routledge & Kegan Paul Ltd.
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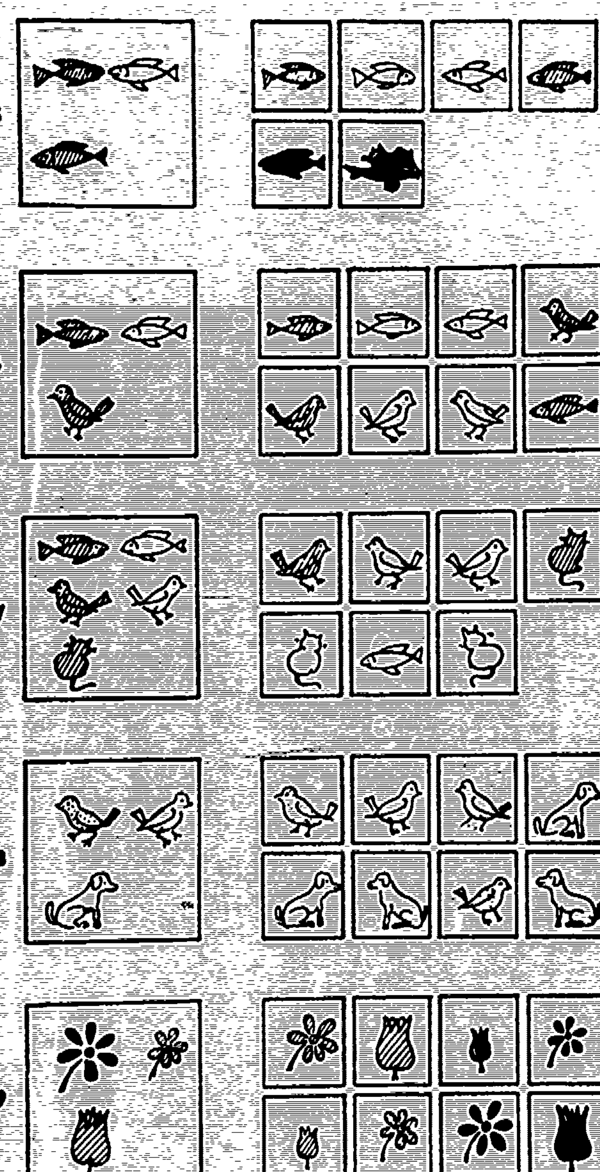
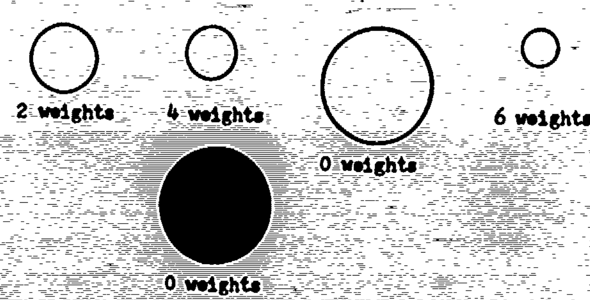


Fig. 6—*Cont.*

Figure 7

CONSERVATION OF WEIGHT

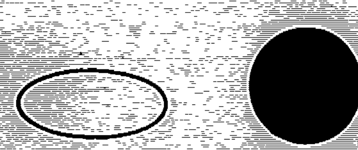
Orientation



CAN YOU FIND OUT WHICH RED BALL WEIGHS THE SAME AMOUNT AS YOUR BLUE BALL?

Test Section

A.



DOES MINE WEIGH AS MUCH AS YOURS OR DOES ONE OF THEM WEIGH MORE?

B.

DO THESE PIECES ALTOGETHER WEIGH MORE, LESS OR THE SAME AS THAT PIECE?

DOES MINE WEIGH AS MUCH AS YOURS OR DOES ONT OF THEM WEIGH MORE?

C.

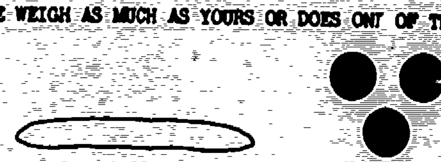


Fig. 7

level in each of the items in a set of tasks, and then considering the evidence for the entire set. Eventually we arrived at a simple 0 (not operational) or 1 (operational) for each of the kinds of tasks: conservation of number and of weight; class inclusion, seriation, reordering, and ordination; and transitivity.

When the data are reduced in this fashion, the richness of the various stages described by the Genevans is lost. On the other hand, such reduction imposes a rigorous test for the hypothesis that operational thinking can be facilitated by curricular intervention.

Before looking at the results as they apply to the various curricula, the overall performance of the children for all programs combined can be examined. Figure 8 shows, for each of the seven kinds of posttest tasks, the proportion of children with clearly operational performance.

The tasks represent three different kinds of operations that are essential for logical thinking: conserving, classifying, and serial ordering. For each operation Piaget has shown a typical sequence of development, into which the tasks included here might be expected to fall. Thus children would be expected to conserve number and quantity before they would be able to conserve weight, to arrange a series before they could solve the problem

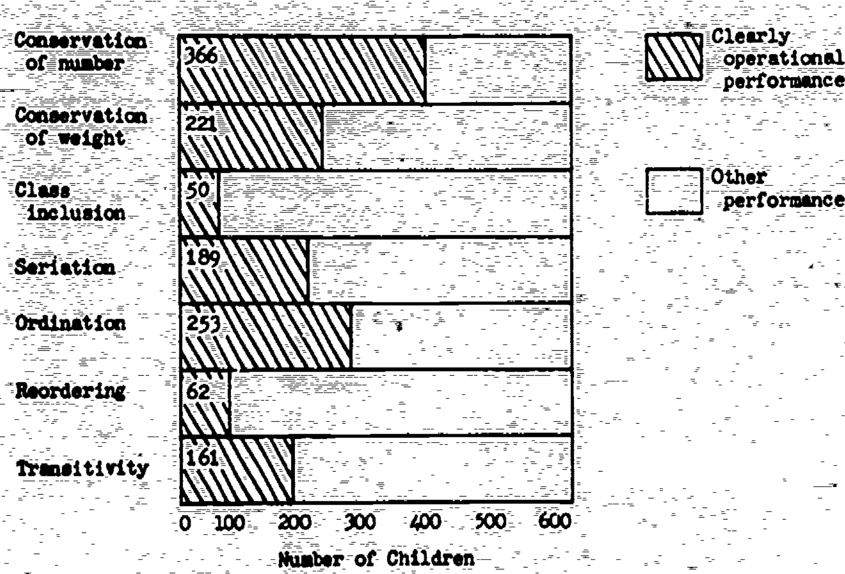


Fig. 1. Proportion of 629 second grade children with clearly operational performance in seven kinds of post-test tasks.

Fig. 8

of ordination, and to accomplish both of these before they could reorder a series that has been scrambled. The transitivity problem may be regarded as a problem of the seriation type, successful solution following seriation and ordination and possibly reordering. The transitivity problem may also be regarded as an extension of the conservation problems involving the conservation of length. Successful solution of the inclusion problems represents the most advanced sort of classification.

On a theoretical basis, then, one might expect to find an ordering of the conservation and seriation kinds of tasks, then be able on an empirical basis to combine these orders with each other and with the class-inclusion task in such a way that the performance of the children could be classified according to predominant patterns.

For seven kinds of tasks, each kind receiving a score of 0 or 1, 128 patterns of performance are possible. Actually, among the children in the study for whom complete data were available,[1] 78 different patterns appeared. Of these patterns, the 1 1 1 1 1 1 1 (clear evidence of logical thinking in all seven kinds of tasks) appeared only three times, and the pattern 0 0 0 0 0 0 0 (no evidence of operational thinking) appeared 136 times. Three other frequent patterns were 1 0 0 0 0 0 0 (operational only in conservation of number and quantity), shown by 55 children; 0 0 0 0 1 0 0 (operational only in ordination), shown by 31 children; and 1 1 0 0 0 0 0 (operational in both conservation of number and quantity and weight), shown by 26 children. No other patterns appeared as frequently as these.

In 314 cases the patterns conformed to what might have been expected on the basis of the sequences described above. In the remainder of the patterns children were scored operational in theoretically more difficult tasks when they had not been scored as operational in less difficult tasks. Attempts to order them, either on the basis of theory or empirically, were unsuccessful.

Considering the narrow age range of the group, the general unreliability of performance of children under the age of eight, and the inclusion of four kinds of operations in the patterns, the expectation of ordering them was, perhaps, unwarranted. The intercorrelations (phi coefficients) among the tasks were low, the largest being 0.32 between conservation of number and conservation of weight. This suggests that for seven-year-olds the prediction of an operational response from one task to another is hazardous even when the tasks bear as close a theoretical relationship to one another as do the two conservation tasks or the seriation and ordination tasks.

1. The reader will note that the total number of children varies in some of the tables. This is because some protocols had missing information and could not be used in all of the analyses.

With the overall performance of the children in view, we can now turn to the results so far as curricular intervention is concerned. They are somewhat paradoxical. On the one hand, when the performances in the posttests of the children who received prescribed lessons beginning in kindergarten are compared with the performances of those who did not receive instruction until first grade, a number of significant differences are apparent, taking PPVT I.Q. into account. Here it must be remembered that little was known about the nature of the kindergarten experience of the children who did not begin prescribed lessons until first grade.

In contrast, when the performances of the children who had prescribed lessons beginning in kindergarten are compared with the performances of the group about whose kindergarten experience information is available but who did not have prescribed lessons, we see that the latter do as well as their counterparts who had the lessons. Table 2, which shows matched groups drawn from each of the four programs, highlights this finding.

Several resolutions of the paradoxical findings are suggested in the published report of the study. The one I lean toward says that the intro-

TABLE 2

Number of Children in Four Instructional Programs, Matched for Sex, C.A.,[a] I.Q.[b] and Initial Status in Conservation,[c] Giving Operational Responses in Post-Test Tasks

Program	N	Conservation of No.	Conservation of Wt.	Class Inclusion	Serial Ordering: Seriation	Serial Ordering: Ordination	Serial Ordering: Reordering	Transitivity
		Initiated in Kindergarten						
AAAS (CCMP)	26	19	15	2	6	12	1	14
SCIS (CCMP)	26	20	13	2	9	12	3	4
CCMP Only	26	10	8	2	9	12	4	4
No Prescribed Lessons	26	19	13	1	11	15	2	9

[a]Within three-month range.

[b]Within six-point range on Peabody Picture Vocabulary Test, Form A.

[c]Number of tasks correct.

duction of mathematical and scientific concepts in the curriculum of early childhood can facilitate the development of logical thinking. Such introduction is more efficacious when it occurs in kindergarten than when it is postponed to first grade. However, experienced teachers familiar with and interested in the elementary concepts, planning activities around the children's interests, are as effective as teachers who use prescribed lessons. (I think this is in line with the analysis of training studies that Beilin has made here, suggesting that a number of kinds of training may, under certain circumstances, be effective.) We don't have sufficient data to prove that this is the correct resolution, but it seems to make sense in the context of the development of logical thinking that we have inferred from the results.

Piaget has postulated an integration of the abilities involved in classifying, ordering, and conserving at about the age of seven, when the child becomes truly operational:

> It is remarkable to see the formation of a whole series of these groupings by children at about age seven. They transform the intuitions into operations of all kinds and explain the transformation of thinking described earlier. Above all, it is striking to see how these groupings are formed, often very rapidly, through a sort of total reorganization. No operation exists in an isolated state; it is always formed as a function of the totality of operations of the same kind. For example, a logical concept of class (combination of individuals) is not constructed in an isolated state but necessarily within a classification of the grouping of which it forms a part. A logical family relation (brother, uncle, etc.) is constituted only as a function of a set of analogous relations whose totality constitutes a system of relationships. Numbers do not appear independently of each other (3, 10, 2, 5, etc.); they are grasped only as elements within an ordered series: 1,2,3 . . . , etc. Likewise, values exist only as a function of a total system or "scale of values." An asymmetric relationship such as $B < C$ is intelligible only in relation to the possible seriation of the set $0 < A < B < C < D \ldots$, etc. What is even more remarkable is that these systems of sets are formed only in the child's thinking in connection with the precise reversibility of the operations, so that they acquire a definite and complete structure right away. [Piaget 1967, p. 49]

The low correlations among the performances in the various tasks in our study simply do not support the idea of integration at age seven. This may be an artifact of the standardized procedure, and of the inclusion of children who, although they were in second grade, had not yet reached the age of seven. Nevertheless, the range of tasks presented and the number of children involved are sufficient to warrant some generalization. The child beginning second grade is typically inconsistent and often illogical

when confronting the kinds of tasks posed in the study. He has not yet developed, or does not readily bring to bear on the tasks, the coordinated structures that Piaget describes as typical of the child who has reached the operational level of thinking.[2]

The issue here, of course, is not so much the timing of the transition as the question of the extent of reorganization of the child's thinking, how it is brought about, and most importantly how education contributes to it. Without clinical or qualitative appraisal of the individual it may be difficult to grasp the "functional unity" that Piaget and Inhelder (1969) describe (within each subperiod) as binding "cognitive, playful, affective, social and moral reactions into a whole" (p. 128).

Further, without longitudinal appraisal, and appraisal in the classroom setting, it is equally difficult to judge the extent to which the child's educational encounters do indeed contribute to his thinking.

The clinical approach has rarely been taken by American investigators, largely because of fear that the child's responses might reflect the interviewer's biases more than his own convictions, but partly, no doubt, because the clinical interview takes so much time. In this regard we have much to learn from our British colleagues, notably among them K. Lovell. Their interview schedules, while sufficiently standardized for reliable replication, also allow for the probing necessary to reveal the suspect answers. For some purposes, and particularly for training teachers, many opportunities to explore children's thinking in a clinical way are imperative.

Under certain circumstances, however, the standard procedure has some advantages. For example, in the longitudinal study the repetition of the same problems and the same questions tends to highlight the changes in the children's responses. Given a sufficient number of children, the probabilities of change from one level of thinking to another at successive interviews can be calculated. As our second longitudinal study shows, the rate of such transition can be used as a means of comparison of the effectiveness of different curricular interventions.

One of the hazards of the successive repetition of standard interviews is that the original problems and the questions may not be as effective as one anticipates. In our first longitudinal study, for example, for the child who did not give a spontaneous response to "What about now?" we consistently used the question "Are there more here, more here, or are they the same?" This standard procedure was easy for the interviewers, but some false positives might have been avoided if the question had varied to "Are

2. This paragraph was written before reading *The Development of the Concept of Space in the Child* by Laurendeau and Pinard, wherein Piaget comments on the general problem of *décalages*.

they the same, or are there more here, or more here?" Similarly, "How can you tell?" or "Explain that to me" have proved to be more neutral than the "Why?" or "Why do you think so?" questions, which some children take to mean that they are in error.

Attrition constitutes another and perhaps the most obvious hazard to longitudinal research. In the first longitudinal study over 20 percent of the children studied in kindergarten were no longer available at the end of first grade. In the second study the loss from kindergarten to the beginning of second grade was nearly 35 percent. In many instances the children were still available in the school system but had moved away from the school that was using the innovative curriculum. The rate of attrition may be somewhat less when kindergarten, which parents sometimes regard as less critical to the child's schooling, is not included.

All of this suggests that from a practical viewpoint three years is about as long as one can hope to keep a large-scale longitudinal study going. Data for youngsters followed from kindergarten or earlier through the period of concrete operations and into formal operational thinking would be extremely useful. However, except for one study reported by the Geneva group, but apparently as yet unpublished, such data do not seem to be forthcoming. A possible substitute might be three parallel three-year studies encompassing ages five through seven, seven through nine, and nine through eleven. If an analysis of the data of the overlapping groups indicated them to be comparable, the data might be combined and the sequences and rates of change derived, as from a six-year longitudinal study.

Considering the number of rather large-scale cross-sectional studies dealing with a variety of kinds of thinking—as, for example, those in mathematics under sponsorship of Professor Rosskopf, Professor Lovell's studies in England, and those he has inspired here, such as Lillian Whyte's study of classification among Canadian children with normal and subnormal mental ability—the necessity for *large-scale* longitudinal studies seems not very great.

What appears to be more needed at this point is smaller-scale longitudinal studies. Some might well be sets of case studies that include repeated interviews and also include observational material related to classroom functioning.

Piaget has sketched the grand design of developing intelligence. He and his followers have now filled in many of the details of the ways knowing proceeds from infancy to adolescence. Innumerable researchers, including many who, on various theoretical grounds, contest the adequacy of the Piagetian view, have explored and manipulated many aspects of children's thinking. But it is not Piaget's idea that a few sessions with an

experimenter, or thirty minutes a day with a new mathematics program, is likely to importantly change a child's way of thinking. The success of any intervention into the child's thinking, either experimental or curricular, can be measured only by its pervasive and durable effects.

Proponents of the modern British primary school have, it seems, understood Piaget in this regard considerably better than have the Americans. Yet they too are only beginning to come to grips with the implications of the theory for the education of the child. In exploring possibilities, however, it seems they have been much more inclined than we to go directly to classrooms and to *involve teachers in the exploration.*

Active participation of teachers in planning, carrying out, and evaluating research in the development of children's thinking seems imperative. It is true that not all teachers will be willing to participate in such an active fashion, and this will limit the generalizations to be made. But as our own studies bear witness, it is clearly preferable to experiment in classrooms where the teacher is truly involved than to draw conclusions from situations in which the teacher's participation is minimal or grudging.

Suppose that a longitudinal study of at least a school year in length is planned. All the children in a given class might be individually interviewed, with the focus perhaps on a single, but pivotal, operation—conservation is clearly a likely choice. Or a sample of children might be more exhaustively interviewed. In either event such interviewing would be either shared by or reported to the teacher, and the implications of the results for the planned curriculum discussed. Several of the children could then be systematically observed and interviewed as the year proceeded, in all instances with the teacher sharing the resulting data. Possibly such systematic study of a relatively small group of youngsters could also yield data about individual styles of coping with problems and about preferred ways of learning. Year-end interviews with all the children, as well as with those intensively studied, would provide information on the possible effects of such study as well as on the general progress of the group.

A plan of this sort could be used to investigate a single aspect of the curriculum—mathematics, for example—or, more narrowly, a single topic; or it might deal more broadly with the child's thinking in various aspects of the curriculum. For example, one ought to be able to see the influence of mathematics in the children's performance in the science program.

Some proponents of Piaget's theory—Hans Furth, particularly—hold that the traditional three Rs curriculum for the early elementary school should be abandoned in favor of a curriculum for thinking—the specific content in the years before the fifth grade being of considerably less importance than the opportunities provided for the child's knowing. Such a

proposal seems to exemplify the "functional unity" of the child, described by Piaget, better than do attempts to tie segmented aspects of the curriculum to the sequences of development described by Piaget. But there is no necessity to take a particular stance on the matter while investigating the actual manifestations of children's thinking in classrooms with varied kinds of curricula. The only danger is that the investigator may find himself working in situations where there is truly very little provocation for thinking.

Another hazard in the observation of the child in the classroom is, of course, the difficulty involved in inferring the child's thinking from his behavior. Language, from the viewpoint of the Piagetian, may mask as well as reveal thought. But, as the work of Dr. Sinclair and others has shown, the lexical quality of the language does provide some clue to the structure of thought. Furthermore, from a practical point-of-view the teacher has no alternative to inferring the child's knowing from what he says and does. While much remains to be known about the complexities of cognitive interaction in the classroom, it can be recorded and analyzed, as the pioneer work of investigators such as B. O. Smith, A. Bellack, and Hilda Taba has demonstrated. In our own study Felice Gordis has shown that such analysis can be done so that the categories parallel those that are applicable in the Piagetian interview.

Piagetian interviews, repeated at intervals, suggest something of the process of change underlying the child's thinking. But they do not reveal the dynamic processes underlying such change. If the insights provided by Piaget's theory are to be integrated into a theory of education, an essential next step appears to be the intensive developmental study of individual children in the natural setting of their classrooms.

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Invited Participants Attending

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Millie Almy
Otto C. Bassler
Harry Beilin
Meryl Blumenthal
Nicholas Branca
Arthur Coxford
Gene Craven
Edward Davis
Peter C. Dodwell
Sister Elinor Ford
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Glenadine Gibb
Daisy Gogan
Graves Halford
Bruce Harrison
Larry L. Hatfield
Maureen Herman
Janellen Huttenlocher
Judith E. Jacobs
David Johnson
Martin L. Johnson
Jeremy Kilpatrick
Richard Kratzer
Charles Lamb
Richard Lesh
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Hermine Sinclair
Akiva Skidell
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Leslie Steffe
Earl Swank
Stanley Taback
Thalia Taloumis
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Leland F. Webb
Lucian Wernick
Harold J. Williford
James W. Wilson

Observer

Mary Kohlerman, representing the National Science Foundation